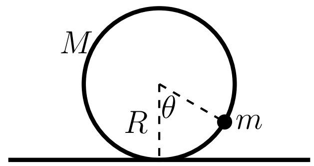
Problem I.1

A cylindrical hoop of zero thickness, mass M, and radius R has a point mass m attached to its circumference. The hoop rolls without sliding along a horizontal plane under the influence of gravity (the gravitational acceleration is g).



(a) **[5 points]** Show that the kinetic energy of the system is

$$K = MR^2\dot{\theta}^2 + mR^2\dot{\theta}^2(1-\cos\theta), \tag{1}$$

where θ is the angle between the vertical direction and the line connecting the center of the hoop to the mass m.

(b) **[5 points]** Obtain the Lagrangian of the system and the equation of motion for the variable θ.

(c) **[5 points]** Keeping terms up to order θ and $\dot{\theta}$ in the equation of motion, determine the frequency ω of small oscillations around the equilibrium position. What is the behavior of ω in the small M limit?

(d) **[5 points]** Return to part (b) and explicitly set $M = 0$. What is the equation of motion for small oscillations in this case? How does the frequency depend on the amplitude θ_0?

(e) **[5 points]** The frequency obtained in the limit $M \to 0$ in parts (c) and (d) are different. Examine the implicit assumptions made in each case to reconcile this apparent discrepancy.

Problem I.2

In the so-called "Lorentz model" of electromagnetic wave propagation in gases, dielectrics, conductors and plasmas, bound electrons are approximately described as a collection of uncoupled harmonic oscillators driven by an external electric field $\mathbf{E}$. The binding frequency of the oscillator is Ω_B, and the mean electron density is N atoms/cm^3. Ignoring magnetic effects, the corresponding harmonic oscillator model equation is

$$\left(\frac{\partial^2}{\partial t^2} + \Omega_B^2\right)\delta\mathbf{x} = \frac{q}{m}\,\mathbf{E}(\mathbf{r}, t), \tag{1}$$

where $\delta\mathbf{x}$ denotes the electron's displacement from equilibrium.

(a) [**5 points**] Laser light with frequency ω in the optical regime propagates in the medium. Assuming the wavelength is long compared to atomic dimensions, use Eq. 1 to write an expression for the electric susceptibility $\chi(\omega)$, which relates the polarization and electric fields via $P = \epsilon_0 \chi E$.

(b) [**5 points**] Write the expression for the permittivity (dielectric constant) of the medium at the laser field frequency.

(c) [**4 points**] Sketch the permittivity $\epsilon(\omega)$ as a function of ω near the binding frequency Ω_B. Using this dielectric constant in the dispersion relation for electromagnetic waves, describe the physical consequences for wave propagation in the distinct regimes you see.

(d) [**4 points**] In the case of a free electron, i.e., $\Omega_B = 0$, calculate the critical "plasma" frequency ω_p for which $\epsilon(\omega_p) = 0$. Give numerical values for ω_p for metals where $N = 10^{28}$ electrons/m^3 and plasmas in the ionosphere where $N = 10^{12}$ electrons/m^3. Use $\epsilon_0 \approx 10^{-11}$ F/m, $m \approx 10^{-30}$ kg, and $q \approx 10^{-19}$ C. Compare your results to the frequency of visible light and comment on the consequences for incident light from the sun.

(e) [**3 points**] Calculate the frequency-dependent group velocity $v_g(\omega)$.

(f) [**4 points**] Estimate the maximum value of electric field beyond which this semi-classical model breaks down. Your answer should be in terms of electronic charge q, the electron mass m, the electronic binding frequency Ω_B and the atomic lengthscale (Bohr radius, a_B).

Problem I.3

A three-dimensional cubic lattice of spin-1/2 atoms is a model of a ferromagnet. A simplified Hamiltonian for this system can be written as

$$H = -J \sum_{\langle i,j \rangle} \sigma_i \sigma_j - \mu B_{ext} \sum_i \sigma_i, \tag{1}$$

in which σ_i may assume only the values ± 1, and the summation condition, $\langle i,j \rangle$, restricts the sum to nearest neighbor pairs. The "exchange constant" J has dimensions of energy, and B_{ext} is an externally applied magnetic field that interacts with the magnetic moment μ associated with the spin.

(a) [**4 points**] Convert Eq. 1 into a simplified "mean field" Hamiltonian by assuming all neighboring σ_js are replaced by their mean or average value $\bar{\sigma}$ (statistical mechanical or thermal average):

$$H_{MF} = -\mu \sum_i \sigma_i (\bar{B} + B_{ext}), \tag{2}$$

What is $\bar{B}$ in terms of J, μ, and $\bar{\sigma}$? What is the magnetization per site M in terms of $\bar{\sigma}$?

(b) [**3 points**] Using the mean-field Hamiltonian H_{MF} and the Boltzmann distribution for the probabilities of $\sigma_i = \pm 1$, calculate the magnetization M as a function of $\bar{B} + B_{ext}$, $M = F(\bar{B} + B_{ext})$.

(c) [**3 points**] Since $\bar{B}$ is proportional to M the equation $M = F(\bar{B} + B_{ext})$ is referred to as the self-consistency equation for the mean-field theory. Using the answer to part (b) and assuming $B_{ext} = 0$, so that $M \equiv F(\bar{B})$ can be considered to be a function of $\bar{B} \propto M$, show that $\bar{B} = 0$ is always a solution to the self-consistency equation.

(At sufficiently large temperatures, this is the only real solution and is referred to as the "paramagnetic" state, since it is the state with zero magnetization at zero external field.)

(d) [**5 points**] Assume $B_{ext} = 0$ for this part so that $M = F(\bar{B})$. By expanding the right hand side of this equation to order $O(\bar{B}^3 \propto M^3)$ show that there is more than one real solution below a critical temperature T_c. What is T_c?

(The state with a non-zero magnetization at zero external field is referred to as the "ferromagnetic" state.)

(e) [**5 points**] In the paramagnetic phase, compute the zero-field "mean-field" susceptibility i.e. $\chi_{pm} = \partial_{B_{ext}} F(\bar{B} + B_{ext})|_{B_{ext}=0}$. In general χ is a measure of the response of the spin system to an external field. What is the behavior of χ_{pm} as $T \to T_c$ from above? How does this result show that the paramagnetic state becomes unstable at T_c? The temperature T_c (calculated in (d)) is the transition temperature between the paramagnetic and ferromagnetic state, i.e. the "Curie temperature".

Problem I.4

π^0 mesons can be produced via the following type of reaction:

$$A + B \to A + B + \pi^0.$$

A study was performed where the incoming projectile, A, was a He^3 particle ($m_A \simeq 3M_N$). The stationary target, B, was C^{12} ($m_B \simeq 12M_N$) or Pb^{208} ($m_B \simeq 208M_N$). Here $M_N = 940\mathrm{MeV}/c^2$. (Below we use units with $c = 1$.)

(a) [**13 points**] Obtain a relativistically correct expression for the projectile's threshold kinetic energy, required to produce the π^0 meson, in terms of m_A, m_B and m_π.
If the He^3 incident energy is 160MeV in the lab frame, show that π^0 mesons ($m_\pi = 0.14M_N = 135\mathrm{MeV}$) can be produced only with the Pb^{208} target. Consider the projectile and target as elementary particles.

Now suppose that a higher-energy He^3 projectile is available. Use your calculation from part (a) to answer the following:

(b) [**12 points**] The π^0 mesons decay into two γ-rays with a *proper* half-life of $\tau = 2 \times 10^{-16}$ s. At the threshold energy required for π^0 production via the C^{12} target, determine the (lab frame) distance the mesons will travel in vacuum before half of them decay. Numerical results should have units and need only be accurate to 15%. Indicate why any approximations you make are of this 15% accuracy.

Problem I.5

The equations of particle number and momentum conservation of a gas are

$$\partial\rho/\partial t + \nabla \cdot (\rho \boldsymbol{v}) = 0,$$

$$\partial \boldsymbol{v}/\partial t + \boldsymbol{v} \cdot \nabla \boldsymbol{v} = -\frac{\nabla p}{m\rho},$$

where m is the mass of a gas particle, $\rho(\boldsymbol{x}, t)$ is the particle number density, $\boldsymbol{v}(\boldsymbol{x}, t)$ is the velocity field, and $p(\boldsymbol{x}, t)$ is the gas pressure. We assume that the gas is ideal so that

$$p = \rho K_B T,$$

where $T(\boldsymbol{x}, t)$ is the temperature and K_B is Boltzmann's constant. (Viscosity has been neglected in the above.)

(a) [**7 points**] Linearize these equations for small perturbations from a no-flow ($\boldsymbol{v}_0 = 0$), time stationary, spatially homogeneous state, assuming that $\rho(\boldsymbol{x}, t) = \rho_0 + \rho_1(\boldsymbol{x}, t)$, $\boldsymbol{v} = \boldsymbol{v}_1(\boldsymbol{x}, t)$, $p(\boldsymbol{x}, t) = p_0 + p_1(\boldsymbol{x}, t)$ and $T(\boldsymbol{x}, t) = T_0 + T_1(\boldsymbol{x}, t)$, where the quantities with the subscript 1 are the small perturbations. Write the relations implied between the perturbation quantities (ρ_1, $\boldsymbol{v}_1$, p_1, and T_1).

(b) [**6 points**] Isaac Newton in his consideration of sound essentially assumed that $T_1 = 0$. Show that, with this assumption, the perturbed density ρ_1 satisfies a wave equation of the form

$$\nabla^2 \rho_1 - \frac{1}{C_N^2}\frac{\partial^2 \rho_1}{\partial t^2} = 0,$$

and express Newton's speed of sound C_N in terms of the gas parameters.

(c) [**6 points**] Newton's theoretical result for the speed of sound stood for about 125 years, although experiments in ambient air clearly showed that it was too small. Laplace finally reconciled this contradiction by hypothesizing that compression and rarefraction of ambient air by audible sound waves was an adiabatic process,

$$p/\rho^\gamma = \text{(constant)},$$

where γ is the usual ratio of specific heats. Show that this adiabatic hypothesis again yields an equation of the form in part (b), but now with a different expression (give it) for the sound speed. Denote this new expression for the sound speed by C_L (where the subscript L stands for Laplace). (C_L agrees well with experiments in air.)

(d) [**3 points**] For air at room temperature, γ (the ratio of specific heats) is approximately 7/5. What is the ratio (C_N/C_L) by which Newton's result underestimated the sound speed in air?

(e) [**3 points**] Let ξ denote the thermal diffusivity of the gas (units of $\xi = [\text{length}]^2/[\text{time}]$). Considering sinusoidal sound waves of period τ and wavelength λ, under what condition on ξ would Laplace's adiabatic hypothesis be valid? Explain. Under what condition on ξ would Newton's isothermal assumption be valid?