UNIVERSITY OF MARYLAND
Department of Physics
College Park, Maryland

PHYSICS Ph.D. QUALIFYING EXAMINATION
PART II

August 28, 2015 9:00 a.m. – 1:00 p.m.

Do any four problems. Each problem is worth 25 points. Start each problem on a new sheet of paper (because different faculty members will be grading each problem in parallel).

Be sure to write your Qualifier ID ("control number") at the top of each sheet — not your name! — and turn in solutions to four problems only. (If five solutions are turned in, we will only grade # 1 - # 4.) For whichever problem (or problems) you skip, fill in a placeholder form so that the grader for that problem will have something from every student.

You may keep this packet with the questions after the exam.

Problem II.1

(a) **[6 points]** Consider a one-dimensional simple harmonic oscillator: a particle with mass m bound to a quadratic potential $V = \frac{1}{2}kx^2$. Show by explicit calculation that

$$\langle u_n|\frac{1}{2m}p^2|u_n\rangle = \langle u_n|\frac{1}{2}kx^2|u_n\rangle$$

holds for the eigenstates $|u_n\rangle$, which are normalized in the usual way.

(b) **[6 points]** Write down the Hamiltonian for a system in which there are two non-identical particles (of mass m_1 and m_2) moving in one dimension and interacting via a quadratic potential $V = \frac{1}{2}k(x_2 - x_1)^2$, where x_1 is the position of the first particle and x_2 is the position of the second particle. Find the energy of the ground state and its wavefunction in coordinate space.

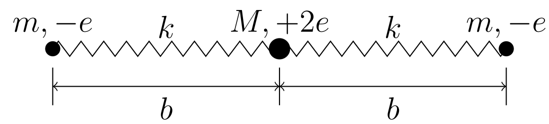
(c) **[7 points]** Explain how the energy and wavefunction of the ground state would be modified if the system in part (b) was composed of two *identical* spin $\frac{1}{2}$ particles.

(d) **[6 points]** Now add a spin-spin interaction between the identical spin $\frac{1}{2}$ particles given by $\alpha(x_2 - x_1)^4 \boldsymbol{\sigma_1} \cdot \boldsymbol{\sigma_2}$, where α is a small positive constant and $\boldsymbol{\sigma} = \sigma_x \hat{\boldsymbol{x}} + \sigma_y \hat{\boldsymbol{y}} + \sigma_z \hat{\boldsymbol{z}}$. Using perturbation theory, find the energy of the ground state to first order in α.

Possibly useful information: $\int_{-\infty}^{\infty} y^4 e^{-y^2} dx = 3\sqrt{\pi}/4$.
$a^\dagger = \sqrt{\frac{m\omega}{2\hbar}}(x - \frac{ip}{m\omega})$
$a = \sqrt{\frac{m\omega}{2\hbar}}(x + \frac{ip}{m\omega})$

Problem II.2

Consider a linear triatomic molecule with ions constrained to move only along the axis of the molecule (which can be taken as the x-axis). The molecule consists of two ions of mass m and charge $-e$ that are symmetrically located at either side of an ion of mass $M = 2m$ and charge $+2e$ at equilibrium. The complicated forces between the ions are approximated by two springs with spring constant k, and equilibrium length b, as shown in the figure.



(a) **[2 points]** Write down the Hamiltonian for this system of three coupled masses.

(b) **[4 points]** Show that the following transformation

$$
\begin{aligned}
x_1 &= \frac{1}{2}(Q_B - \sqrt{2}Q_A) \\
x_2 &= -\frac{1}{2}Q_B \\
x_3 &= \frac{1}{2}(Q_B + \sqrt{2}Q_A)
\end{aligned}
$$

simplifies both the kinetic and the potential energies and eliminates the center-of-mass motion.

(c) **[8 points]** Q_A and Q_B are normal-mode coordinates describing the internal vibrations of the molecule. These modes satisfy

$$Q_j(t) = Q_j(0)e^{i\omega_j t}, \quad j = (A, B).$$

Describe the motion of the ions in each of the normal modes. Determine the oscillation frequencies ω_j and the electric dipole moments D_j for the two internal modes.

The molecule is actually a quantum mechanical system. Initially it is in its ground state; it is then subjected to a *weak* uniform electric field $E_x(t) = E_0(\omega)\cos(\omega t)$. The perturbing interaction between its dipole moment and the electric field is $H' = -DE_x$, where D is the electric dipole moment and E_x is the electric field component along the molecule. The transition probabilities $P_j(\omega, t)$ for excitation of states with one quantum either in mode A or in mode B is

$$P_{0j}(\omega, t) = \frac{|V_{0j}|^2}{\hbar^2}\frac{\sin^2[(\omega_{0j} - \omega)t/2]}{(\omega_{0j} - \omega)^2},$$

where $|V_{0j}|^2$ is the matrix element of the perturbing Hamiltonian, and in this case is $|V_{0j}|^2 = |D_j E_0(\omega)|^2$ for mode j of frequency ω_{0j}.

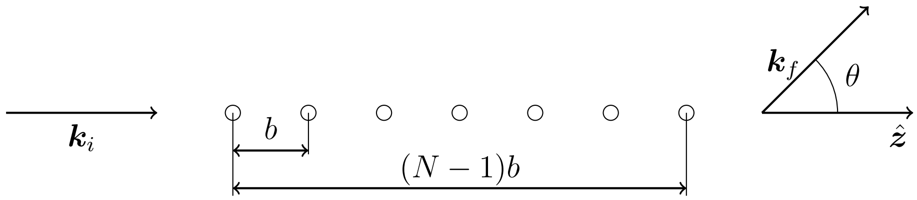
(d) **[3 points]** Use the equation above to obtain expressions for $P_{0A}(\omega, t)$ and $P_{0B}(\omega, t)$.

(e) **[8 points]** Determine the transition rates $R_j = \frac{d}{dt}\int d\omega P_{0j}(\omega, t)$, for a uniform incoherent beam of radiation propagating in the z-direction and polarized along x, whose intensity per frequency interval is given by $J(\omega) = \frac{1}{2}c\epsilon_0 |\ E_0(\omega)\ |^2$, where c is the speed of light in vacuum and ϵ_0 is the permittivity of free space. Assume that $J(\omega)$ depends weakly on frequency.

Possibly useful information: $\int\limits_{-\infty}^{\infty} \frac{sin^2 x}{x^2} dx = \pi$.

Problem II.3

Consider diffraction of quantum particles of wavevector $\boldsymbol{k}_i$ on a system of N identical scatterers located a distance b apart along the direction of incidence $\hat{\boldsymbol{z}}$, as shown in the figure below.



Each individual scatterer is characterized by a potential $U_0(r)$ and the corresponding Born amplitude of scattering $f_0(\theta)$, which is assumed to be a smooth, featureless function.

(a) [**3 points**] First consider the case of only one scatterer ($N = 1$). In the Born approximation, express the amplitude $f_0(\theta)$ and the differential cross section $d\sigma_0/d\Omega$ of scattering in terms of $U_0(r)$ and k.

(b) [**6 points**] Now consider N scatterers. The total potential is

$$U(\boldsymbol{r}) = \sum_{n=0}^{N-1} U_0(|\boldsymbol{r} - nb\hat{\boldsymbol{z}}|),$$

where $\hat{\boldsymbol{z}}$ is the unit vector along the z axis.

In the Born approximation, calculate the amplitude $f(\theta)$ and the differential cross section $d\sigma/d\Omega$ of scattering in terms of $f_0(\theta)$, k, b, and N.

Explore the answer obtained in Part (b) for various values of kb as follows:

(c) [**4 points**] What are $f(\theta)$ and $d\sigma/d\Omega$ in the limit $Nbk \ll 1$? Interpret the result.

In the case of $kb \geq 1$ and $N \gg 1$, answer the following questions and give geometrical interpretations of your results:

(d) [**4 points**] Sketch $d\sigma/d\Omega$ as a function of θ. Calculate the angles θ_n at which $d\sigma/d\Omega$ has strong maxima.

(e) [**4 points**] What is the total number of strong maxima for a given value of kb?

(f) [**4 points**] Discuss how the height and the width of a strong maximum depend on $N \gg 1$. The width $\Delta\theta_n = 2\delta\theta_n$ is determined by the angles $\theta_n \pm \delta\theta_n$ where $d\sigma/d\Omega$ vanishes.

Useful formula:

$$\sum_{n=0}^{N-1} a^n = \frac{a^N - 1}{a - 1}.$$

Problem II.4

Two particles interact via a spin-spin Hamiltonian term $A\boldsymbol{S}_1 \cdot \boldsymbol{S}_2$, where A is a positive constant and $\boldsymbol{S}_{1,2}$ are the spin angular momenta of the two particles. Particle 1 has spin 1 and magnetic moment $\mu_\mathbf{1} = -\frac{\mu_\mathbf{B}}{\hbar}\mathbf{S_1}$, whereas particle 2 has spin $\frac{1}{2}$ and zero magnetic moment.

(a) **[6 points]** What are the energy levels of this system and the degree of degeneracy of the levels?

(b) **[8 points]** Write down the energy eigenstates corresponding to the different energy levels in part (a), as linear superpositions of products of single-particle spin states.

Now consider what happens when the system is in a magnetic field of strength B.

(c) **[4 points]** What are the approximate energy eigenstates and eigenvalues if $B \gg \frac{A\hbar^2}{\mu_B}$?

(d) **[7 points]** Sketch the approximate energy eigenvalues as functions of $0 < B \lesssim \frac{A\hbar^2}{\mu_B}$, and label the appropriate states. *Do not neglect the spin-spin interaction term from parts (a) and (b).*

Possibly useful information: $J_\pm|j,m\rangle \;=\; \sqrt{j(j+1) - m(m \pm 1)}|j, m \pm 1\rangle$

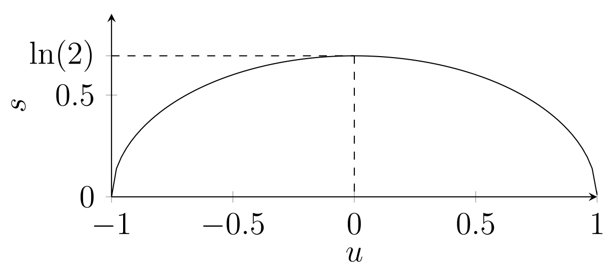
Problem II.5

The notion of negative absolute temperature is unusual but it can occur in, for example, quantum spin systems. To see this, consider a quantum system of N noninteracting magnetic dipoles each with spin 1/2, having a magnetic moment μ_B, placed in a magnetic field $\mathbf{B}$. Assume a canonical ensemble description of this spin system with a temperature $T = 1/(k\beta)$, where k is Boltzmann's constant.

(a) [**3 points**] Write down the energy ϵ of this two-level system in terms of μ_B and B (the magnitude of the magnetic field). Determine the partition function Z_N in terms of $\beta\epsilon$.

(b) [**9 points**] Calculate the free energy F, the entropy S, and the internal energy U of this system as a function of N and β.

A schematic plot of $s \equiv \frac{S}{Nk}$ versus $u \equiv \frac{U}{N\epsilon}$ is provided in the figure below to aid you in answering the following questions:



(c) [**4 points**] What is the temperature T of the magnetic system in terms of quantities you obtained in part (b)? Indicate in the figure: (i) the places where $T = 0$, (ii) the region where negative temperature $T < 0$ appears, (iii) indicate with one arrow on each branch the direction of increasing T, and (iv) what is the temperature at the global maximum of that curve?

(d) [**6 points**] Plot the temperature parameter $\theta = kT/\epsilon$ on the vertical axis versus u (from -1 to $+1$) on the horizontal axis. (i) place an arrow along the curves indicating the directions of decreasing temperature. (ii) Explain what energy state the system is in at $T = 0$. (iii) If a system of this nature at $T = -300K$ somehow interacts with an identical system at $T = 300K$, what is the final equilibrium temperature?

(e) [**3 points**] Name two necessary conditions for a system to manifest negative temperature.