UNIVERSITY OF MARYLAND
Department of Physics
College Park, Maryland

PHYSICS Ph.D. QUALIFYING EXAMINATION
PART II

January 22, 2016 9:00 a.m. – 1:00 p.m.

Do any four problems. Each problem is worth 25 points.
Start each problem on a new sheet of paper (because different faculty members will be grading each problem in parallel).

Be sure to write your Qualifier ID ("control number") at the top of each sheet — not your name! — and turn in solutions to four problems only. (If five solutions are turned in, we will only grade # 1 - # 4.)

At the end of the exam, when you are turning in your papers, please fill in a "no answer" placeholder form for the problem that you skipped, so that the grader for that problem will have something from every student.

You may keep this packet with the questions after the exam.

Problem II.1

At time $t = 0$, the wave function for the hydrogen atom is

$$\psi(\mathbf{r}, 0) = \frac{1}{\sqrt{10}}(2\psi_{100} + \psi_{210} + \sqrt{2}\psi_{211} + \sqrt{3}\psi_{21\bar{1}}),$$

where the subscripts in ψ_{nlm} are the usual energy, angular momentum, and z-projection of the angular momentum quantum numbers with $\bar{1}$ the $m = -1$ state. Ignore spin.

(a) [**5 points**] What is the expectation value for the energy of this system, in eV?

(b) [**5 points**] What is the probability of finding the system with $l = 1$, $m = +1$ as a function of time?

(c) [**5 points**] What is the probability of finding the electron within 10^{-10} cm of the proton at time $t = 0$? (A good approximate result is acceptable.)

(d) [**5 points**] How does this wave function evolve in time; *i.e.*, what is $\psi(\mathbf{r}, t)$?

(e) [**5 points**] Suppose a measurement is made which shows that $n = 2$, $l = 1$ and the x-projection of the angular momentum is $+1$. Describe the wave function immediately after such a measurement in terms of the ψ_{nlm} wave functions used above.

Possibly useful information:

$\psi_{nlm} = R_{nl}Y_{lm}$, $a \approx 0.5$Åis the Bohr radius, and

$$|R_{10}|^2 = \tfrac{4}{a^3}e^{-2r/a},$$
$$|R_{21}|^2 = \tfrac{r^2}{24a^5}e^{-r/2a},$$

$$L_x = \frac{L_+ + L_-}{2}$$
$$L_\pm Y_{lm} = \sqrt{l(l+1) - m(m \pm 1)}Y_{lm\pm1}$$

Problem II.2

Consider the one-dimensional infinite potential well of length L.

(a) **[4 points]** What are the eigenenergies, E_n, and associated normalized stationary state wave functions, $\psi_n(x)$, of a single electron in this potential?

(b) **[6 points]** Now suppose we have two electrons in the potential and treat them as non-interacting, non-identical particles. What are the lowest three allowed energies of the two-particle system, and their associated wave functions? Identify any degeneracies.

(c) Now suppose we couple the particles via an interaction $V = \lambda\delta(x_1\text{-}x_2)$ where x_1 and x_2 are the coordinates of the particles, $\delta(x)$ is the one-dimensional Dirac delta function, and $\lambda > 0$.

   (i) **[4 points]** Use lowest-order perturbation theory to find the energy shifts of the first excited states resulting from the interaction.

   (ii) **[2 points]** To zeroth order in V what are the normalized stationary state wave-functions associated with the energies in part (i)?

   (iii) **[2 points]** Formulate a condition on the coefficient λ for this perturbation theory to be applicable.

(d) **[4 points]** Now consider that the electrons in (b) are identical fermions, each of spin 1/2. What are the wave functions for the ground state and first excited state of the unperturbed system?

(e) **[3 points]** How does the interaction V change the first excited states when we include the spin and fermion nature of the electrons?

Possibly useful integrals:

$\int_0^\pi d\phi \sin^4\phi = 3\pi/8$

$\int_0^\pi d\phi \sin^2\phi \sin^2 2\phi = \pi/4$

Problem II.3

A hypothetical particle of mass m interacts with the electron in the ground state of a hydrogen atom but not with the proton (which is assumed to be fixed at the origin). It scatters elastically i.e. leaves the atom in the ground state. The theory of this interaction is that it can be approximated by a delta-function potential

$$U(\boldsymbol{r} - \boldsymbol{r}') = \lambda \delta(\boldsymbol{r} - \boldsymbol{r}')$$

in which λ is a constant and $\boldsymbol{r}, \boldsymbol{r}'$ are the coordinates of the particle and the electron respectively. Suppose that the particle passes through the atom sufficiently slowly that it sees an average density of the electron in the ground state.

(a) **[5 points]** Determine the effective potential $U_0(\boldsymbol{r})$ that the hypothetical particle sees.

(b) **[6 points]** Determine the elastic scattering amplitude in the Born approximation. Be sure to define any symbols you introduce.

(c) **[9 points]** Determine the differential and total cross-sections in the Born approximation.

(d) **[5 points]** State the conditions for the applicability of the Born approximation in this case.

Additional information: The ground state wave-function of the hydrogen atom is

$$\psi(r, \theta, \phi) = 2(4\pi)^{-1/2} a_B^{-3/2} e^{-r/a_B},$$

where a_B is the Bohr radius.

Problem II.4

In this problem you will use symmetry to gain insight into how a perturbation, H', modifies the wavefunctions and energy levels of a system. In particular, symmetry will be used to determine selection rules for what states are connected in perturbation theory.

The Hamiltonian for a spinless particle in three spatial dimensions has the following form :

$$H = \frac{p^2}{2m} + V_0(r) + \lambda H'$$

where $H' = V'(r)Y_{k0}(\theta)$ and Y_{k0} is a spherical Harmonic, k is a positive integer and λ is a small parameter; $\lambda V'(r)Y_{k0}(\theta)$ is treated as a perturbation. The unperturbed Hamilton is rotationally invariant and with energies given by E^0_{nl} and wave functions $\psi^0_{n,l,m}(\vec{r}) = \langle \vec{r}|n,l,m\rangle^0 = \psi^0_{n,l}(r)Y_{lm}(\theta,\phi)$.

One selection rule is that $m' = m$, which follows from the fact that H' is azimuthally symmetric. As a direct consequence, m is a good quantum number and the perturbative expression for the energy eigenenstate states assumes the following form

$$|\psi_{nlm}\rangle = \left(|nlm\rangle^0 + \sum_{j=1}^{\infty} \lambda^j \sum_{n'=0}^{\infty} \sum_{l'=m}^{\infty} d^{j;nlm}_{n'l'} |n'l'm\rangle^0 \right)$$

where $|\psi_{nlm}\rangle$ labels the state which at $\lambda = 0$ becomes $|nlm\rangle^0$. The first order coefficients are given by $d^{1;nlm}_{n'l'} = \langle n'l'm|H'|nlm\rangle/(E^0_{nl} - E^0_{n'l'})$.

The energy levels for $|\psi_{nlm}\rangle$ also have an expansion in λ: $E_{nlm} = E^0_{nl} + \sum_{j=1}^{\infty} \lambda^j E^j_{nlm}$ with $E^1_{nlm} = \langle nlm|H'|nlm\rangle$.

(a) [**7 points**] Suppose $k = 3$, i.e. the perturbation is $\lambda V'(r)Y_{30}(\theta)$. Consider the state $|nlm\rangle = |140\rangle$. Find which of the first order coefficients $d^{1;140}_{n'l'}$ vanish for reasons of symmetry.

(b) [**6 points**] Suppose $k = 4$, i.e. the perturbation is $\lambda V'(r)Y_{40}(\theta)$. In this case, certain levels $|nlm\rangle$ have $E^1_{nlm} = 0$; for these levels, the first order perturbation of the energy vanish by symmetry. Identify these levels.

(c) [**6 points**] Suppose $k = 3$, i.e. the perturbation is $\lambda V'(r)Y_{30}(\theta)$. In this case, certain levels $|nlm\rangle$ have $E^1_{nlm} = 0$. Identify these levels.

(d) [**6 points**] Suppose $k = 2$, i.e. the perturbation is $\lambda V'(r)Y_{20}(\theta)$. Suppose further that $E^1_{220} = \epsilon$ where ϵ has dimensions of energy. What is E^1_{222} in terms of ϵ?

Possibly useful information (continued on next page):

H' is a spherical tensor T_{k0} and it has parity $(-1)^k$. The Wigner-Eckert theorem implies that $\langle n'l'm'|T_{k0}|nlm\rangle = \langle n'l'||T_{k0}||nl\rangle\langle lkm0|l'm'\rangle$, where $\langle n'l'||T_k||nl\rangle$ is a reduced matrix element and $\langle lkm0|l'm'\rangle$ is a Clebsch-Gordan coefficient. This implies selection rules since the Clebsch-Gordan coefficients may be zero.

Some Clebsch-Gordan coeffcients $\langle j_1 j_2 m_1 m_2|jm\rangle$:
$\langle 2220|22\rangle = \langle 2202|22\rangle = \sqrt{\frac{2}{7}}$
$\langle 2211|22\rangle = -\sqrt{\frac{3}{7}}$
$\langle 222\,{-1}|21\rangle = \langle 22\,{-1}2|21\rangle = -\sqrt{\frac{1}{5}}$
$\langle 2210|21\rangle = -\langle 2201|21\rangle = -\sqrt{\frac{3}{10}}$
$\langle 222\,{-2}|20\rangle = \langle 22\,{-2}2|20\rangle = -\langle 221\,{-1}|20\rangle = -\langle 22\,{-1}1|20\rangle = \langle 2200|20\rangle = \sqrt{\frac{1}{5}}$

Problem II.5

Before the development of a microscopic theory of superconductivity, Vitaly Ginzburg and Lev Landau relied on a general theory of second order phase transitions to obtain an equation for a complex order parameter, $\Psi(\mathbf{r})$, for the superconducting state. In the presence of a magnetic field, this (Ginzburg-Landau) equation for $\Psi(\mathbf{r})$ has the form:

$$\frac{1}{2m}\left(\mathbf{p} - \frac{q}{c}\mathbf{A}\right)^2 \Psi + \alpha\Psi + \beta\Psi|\Psi|^2 = 0.$$

The normalization condition is chosen so that $|\Psi(\mathbf{r})|^2 = n_\mathrm{s}$, where n_s is the concentration of superconducting electrons. In accordance with the interpretation of $\Psi(\mathbf{r})$ as a wave function of electronic (Cooper) pairs, $q = 2e$ and $m = 2m_\mathrm{e}$. The temperature- and material-dependent parameters satisfy $\alpha < 0$ and $\beta > 0$.
Sufficiently high magnetic fields destroy superconductivity. As the field decreases, superconducting regions begin to nucleate at a certain critical field H_c, and the order parameter, $\Psi(\mathbf{r})$, becomes nonzero. At the onset of superconductivity, the order parameter is small and the cubic term in the Ginzburg-Landau equation can be neglected. This linearized Ginzburg-Landau equation reduces to a Schrödinger-like equation, which admits reasonable (bounded) solutions only at $H < H_\mathrm{c}$.

Consider a semi-infinite superconductor (occupying the region $x > 0$), which is subjected to a perpendicular uniform magnetic field $\mathbf{H} = H\mathbf{z}$. The boundary can be incorporated as the boundary condition $\left.\frac{d\Psi}{dx}\right|_{x=0} = 0$. Work in the gauge $\mathbf{A}(x, y) = (0, Hx, 0)$ and consider a wave function of the form

$$\Psi(x, y) = f(x)e^{iky}. \tag{1}$$

(a) **[9 points]** Using the linearized form of the Ginzburg-Landau equation and the ansatz Eq. (1), obtain an equation for $f(x)$ and compare it with the Schrödinger equation for a particle in some potential $V(x)$. Identify this potential and give an interpretation of lengthscales $x_k = \hbar kc/qH$ and $\xi = \hbar/\sqrt{2m|\alpha|}$, which appear in the equation.

(b) **[10 points]** Consider the effect of the magnetic field on superconductivity in the bulk (i.e., far from the boundary). Argue that the boundary effects can be ignored for $x_k \gg \xi$. Find the critical magnetic field H_c, at which a bounded solution of the equation from part (a) first appears.

Problem continues on next page

(c) Due to the presence of the boundary, the critical magnetic field H_c^* is actually higher than that calculated in part (b) above. To demonstrate this, consider the case $x_k \approx \xi$ and apply the following trick: In the Schrödinger-like equation for $f(x)$, let x range over the entire interval $-\infty < x < \infty$, but adopt the potential $W(x) = V(|x|)$, where $V(x)$ is the potential calculated previously.

(a) **[2 points]** Sketch both $W(x)$ and $V(x)$.

(b) **[2 points]** Argue that the ground state of the fictitious system still satisfies the boundary condition $df/dx = 0$ at $x = 0$, and its lowest eigenvalue is lower than that found in part (b).

(c) **[2 points]** Argue that it implies a *higher* nucleation field H_c^* than that H_c for a sample without boundary.

Thus, nucleation is favored in the presence of the surface. (This remarkable phenomenon was discovered by D. Saint-James and P.-G. de Gennes in 1963).