UNIVERSITY OF MARYLAND
Department of Physics
College Park, Maryland

Ph.D. PHYSICS QUALIFYING EXAMINATION - PART I

August 26, 2010 **9 a.m. - 1 p.m.**

Do any four problems.

Problems I. 1, I. 2, I. 4 and I. 5 are each worth 25 points.

Problem I. 3 Statistical Mechanics is worth 40 points.

Put all answers on your answer sheets.

Be sure your Qualifier ID Number is at the top right corner of each sheet and turn in solutions to four problems only. If five solutions are turned in we will grade # 1 - # 4.

Problem I.1

A bead of mass m slides without friction on a hoop of radius a that rotates with constant angular velocity ω about an axis perpendicular to the plane of the hoop and passing through the edge of the hoop (see the Figure). The problem ignores both friction and gravity. The angle θ serves as the coordinate and measures the displacement of the bead from a fixed diameter on the hoop.

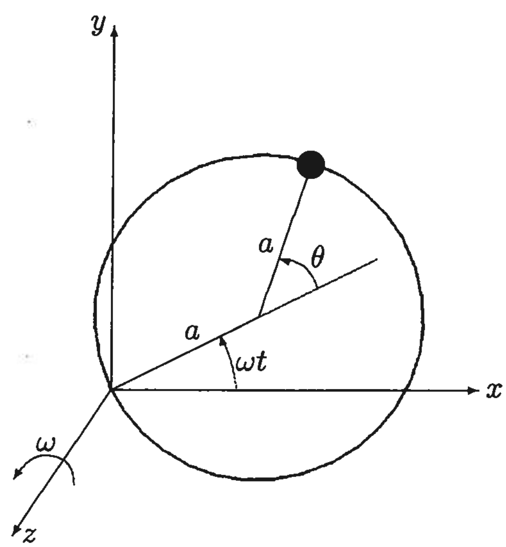
(a) [**5 points**] In terms of the radius a, ω and θ, what are the Cartesian coordinates, $x(t)$ and $y(t)$, of the bead?

(b) [**5 points**] Find the kinetic energy T and the Langrangian L in terms of a, ω and θ. The following identity is useful: $\cos A \cos(A + B) + \sin A \sin(A + B) = \cos B$.

(c) [**5 points**] Derive the equation of motion for the angle $\theta(t)$. Hint: Use the Lagrangian.

(d) [**5 points**] In the small angle approximation, solve for $\theta(t)$. Assume that $\theta(0) = A$ and $\dot{\theta}(0) = 0$. What is the characteristic frequency?

(e) [**5 points**] The Hamiltonian is defined as $H = \dot{\theta} p_\theta - L$, with $p_\theta (= \partial L/\partial \dot{\theta})$ the canonical momentum. Find H in terms of the coordinate and canonical momentum. Is H a constant of the motion?



Problem I.2

In a modern application known as Plasmonics, an important role is played by surface waves. These are electromagnetic waves with fields concentrated close to the interface between vacuum and a medium such as plasma with relative dielectric constant ϵ. The key to the existence of these waves is the negativity of ϵ, for a range of frequencies.

Consider a plane $(y - z)$ interface separating two regions at $x = 0$ (vacuum for $x < 0$, plasma entirely described by a dielectric constant ϵ for $x > 0$). Let z be the propagation direction of a wave along the surface and consider the interplay between the three components of the electromagnetic field: B_y, E_x, and E_z. Assume that all the other components are zero. Assume that there is no variation in the y-direction and assume the time (t) and space dependence of each component is of the form $Re[f(x)\exp(ikz - i\omega t)]$, where $f(x)$ represents the complex amplitude of any of the 3 nonzero fields, k is the wavenumber, and ω is the frequency of the wave.

(a) **[5 points]** Given the space-time dependence as above, write down 3 equations that relate the complex amplitudes of the fields B_y, E_x, and E_z.

(b) **[5 points]** Use these equations to show that complex amplitude $B_y(x)$ satisfies the wave equation $(d^2/dx^2 - k^2 + \epsilon\omega^2/c^2)B_y = 0$, for appropriate ϵ on either side of the interface.

(c) **[5 points]** Find solutions to B_y on either side of the interface such that they are localized near the interface. For given ω and ϵ, what conditions on k must be imposed so that the waves may be localized?

(d) **[5 points]** Write down the boundary conditions that connect B_y and E_z across the two sides of the interface. Suppose $\epsilon < 0$. Use the two boundary conditions to find the wave number k as function of $|\epsilon|$ and ω.

(e) **[5 points]** For a plasma, the dielectric function is given by $\epsilon = 1 - \omega_0^2/\omega^2$, where ω_0 is the plasma frequency. Find ω (eigenfrequency of surface wave) when $k \to \infty$.

Problem I.3

Consider a binary alloy where each site of a lattice is occupied by an atom of type A or B. (A realistic alloy might mix roughly half copper and half zinc to make β−brass.) Let the numbers of the two kinds of atoms be N_A and N_B, with $N_A + N_B = N$. The concentrations are $n_A = N_A/N$ and $n_B = N_B/N$, and the difference is $x = n_A - n_B$. The interaction energies between the neighboring atoms of the types AA, BB, and AB are ε_{AA}, ε_{BB}, and ε_{AB}, correspondingly.

(a) **[4 points]** For a cubic lattice in three dimensions, how many nearest neighbors does each atom have? In the rest of the problem, denote the number of neighbors as c for generality.

(b) **[6 points]** Consider the system at a high enough temperature such that the atoms are randomly distributed among the sites. Calculate the average interaction energy U per site under these conditions. First, express U in terms of n_A and n_B, and then obtain $U(x)$.

In the rest of the problem, consider the case $2\varepsilon_{AB} > \varepsilon_{AA} + \varepsilon_{BB}$ and also assume that $\varepsilon_{AA} = \varepsilon_{BB} = \varepsilon_0$ for simplicity. In this case, sketch a plot of the function $U(x)$ for $-1 \leq x \leq 1$. Indicate locations of the extrema of $U(x)$.

(c) **[6 points]** Under the same conditions (where the atoms are randomly distributed among the sites), calculate the configurational entropy S per site. Assume that $N_A, N_B \gg 1$, so the Stirling approximation $\ln(N!) \approx N \ln N - N$ can be used. First, express S in terms of n_A and n_B, and then obtain $S(x)$.

Sketch a plot of the function $S(x)$. What are the values of S at $x = \pm 1$? For which value of x is the entropy S maximal?

(d) **[6 points]** Using the results of Parts (b) and (c), obtain the free energy per site $F(x,T) = U(x) - TS(x)$, where T is the temperature. Notice that $F(x) = F(-x)$ (because of the assumption $\varepsilon_{AA} = \varepsilon_{BB}$), which simplifies consideration.

Sketch $F(x)$ at a high temperature and at a low temperature. Show that, at a high temperature, $F(x)$ has one global minimum as a function of x. Show that, at a low temperature, $F(x)$ has one local maximum surrounded by two minima, excluding the boundaries at $x = \pm 1$.

(e) **[6 points]** A system tends to minimize its free energy F, subject to externally imposed constraints. A binary alloy with a given x may stay in the uniform state, where the atoms are randomly distributed among the sites, which is called the *mixed* state. However, it may also become unstable with respect to spontaneous segregation into two phases with different values of x, if such a segregation decreases the free energy F. This state is called *unmixed.*

Using $F(x)$ derived in Part (d), show that the uniform mixed state is stable at high temperatures, but becomes unstable below a certain temperature T_*. Determine T_* and the value of x where this instability occurs.

I.3 (Continued)

Hint: The system remains stable as long as $d^2F/dx^2 > 0$ for all x. Determine at what T and x this condition becomes violated.

(f) **[6 points]** For $T < T_*$, the free energy $F(x)$ has two minima at x_1 and x_2. Obtain an equation for $x_1(T)$ and $x_2(T)$. This is a transcendental equation, so you don't need to solve it explicitly for x.

Consider in turn what happens to the binary alloy with a given value x if $x < x_1(T)$, if $x_1(T) < x < x_2(T)$, and if $x_2(T) < x$. Would the state of the binary alloy be mixed or unmixed in these cases? For the unmixed state, what are the values of x in the two phases?

What are the limiting values of $x_1(T)$ and $x_2(T)$ in the limit $T \to 0$? Describe the ground state of a binary alloy at $T = 0$. Does this state minimize the interaction energy U, given that $\varepsilon_0 < \varepsilon_{AB}$?

(g) **[6 points]** For a given x, show that the binary alloy is in the mixed state for $T > T_c(x)$ and in the unmixed state for $T < T_c(x)$. Calculate $T_c(x)$ and sketch it. Indicate the areas corresponding to the mixed and unmixed states on this sketch. Show that T_* is the maximal value of T_c.

Hint: To obtain $T_c(x)$ use the results of Part (f). $T_c(x)$ is obtained from the same equation as $x_1(T)$ and $x_2(T)$.

Problem I.4

Ultra-high-energy cosmic ray protons can lose energy by inelastic collisions with cosmic microwave background (CMB) photons, producing pions: In this problem, we will only look at the reaction $p + \gamma \to p + \pi^0$.

(a) **[10 points]** First, consider this reaction in the reference frame in which the proton is initially at rest. What is the minimum ("threshold") energy, E_t that the photon must have to produce a π^0 by this reaction? (Express your answer in terms of m_p and m_π, the masses of the proton and the pion, respectively.)

Hint: at threshold there is no relative motion between the final state particles, they act like a single particle of mass $m_p + m_\pi$.

(b) **[10 points]** Now, consider the same reaction but viewed in the reference frame of the interstellar medium, in which the proton collides with a CMB photon that has energy E_{CMB}. Assume that the collision is head-on and that $E_{CMB} \ll E_t$, so you may calculate to first order in E_{CMB}/E_t.

What is the energy E_p of the incoming proton in this frame? (Express your answer in terms of the particle masses m_p, m_π, and E_{CMB}.)

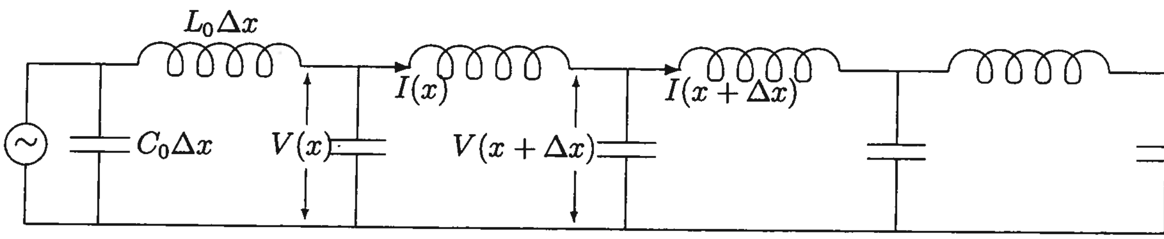
(c) **[2 points]** Using your result from part (b), insert the actual masses of the proton and pion ($m_p = 938$ MeV/c^2, $m_\pi = 135$ MeV/c^2), and a CMB photon energy $E_{CMB} = 10^{-3}$ eV, to calculate E_p in eV.

(d) **[3 points]** This reaction is a way for ultra-high-energy protons to lose energy by colliding with CMB photons, and your result in (c) gives an estimate of the lowest energy (proton energy threshold) at which this energy loss can occur.

Actually, the average energy of a CMB photon is about 6×10^{-4} eV, less than what was used in (c). Explain in one or two sentences why it is appropriate to use the higher CMB photon energy E_{CMB} in calculating this estimate.

If protons with energies higher than the estimated threshold E_p are detected at the Earth, could these protons have come from very far away?

Problem I.5



We will consider electromagnetic waves that propagate on a lossless transmission line that is driven by an ideal alternating current generator. The voltage on the capacitor and current through the inductor as functions of position x and time t in an lossless transmission line (shown schematically above) obey the *Telegrapher's Equations*

$$-C_0\frac{\partial V}{\partial t}=\frac{\partial I}{\partial x} \quad \text{and} \quad -L_0\frac{\partial I}{\partial t}=\frac{\partial V}{\partial x}$$

where C_0 is the capacitance per unit length and L_0 is the inductance per unit length of the transmission line.

(a) [**5 points**] Derive the Telegrapher's Equations by considering how the voltage and current change with a distance Δx along the line.

(b) [**5 points**] Derive a wave equation

$$\frac{\partial^2 V}{\partial x^2}=\frac{1}{v^2}\frac{\partial^2 V}{\partial t^2}$$

from the Telegrapher's Equations and thus find an expression for the speed v of an electromagnetic wave in a transmission line.

(c) [**5 points**] Consider a right-going traveling wave solution to the wave equation for an infinite line in the form $V(x,t)=V_{0+}e^{i(\omega t-kx)}$, $I(x,t)=I_{0+}e^{i(\omega t-kx)}$ and find a relation between ω and k. Calculate the ratio $Z_0=V_{0+}/I_{0+}$, which is called the impedance. How is your answer changed for a wave traveling to the left?

(d) [**5 points**] An ideal ac voltage generator is connected at the origin to a semi-infinite transmission line, which extends from $x=0$ to $x=\infty$. Using the result of part (c), calculate the power leaving the generator and entering the line. (If you had difficulty with part (c), you may assume here that the impedance of the line Z_0 is real.) Given that there are no resistors causing dissipation in this ideal line, what are the implications of a real impedance for the energy flow from the generator?

(e) [**5 points**] A transmission line extends from $x=0$ to $x=\ell$, and a right-going wave with current amplitude I_{0+}, created by the ideal generator at $x=0$, is traveling along the line. The line is terminated at $x=\ell$ as shown in the Figure. Formulate a boundary condition on the current $I(\ell,t)$ at the end of the line and derive a relationship between I_{0-} and I_{0+}, where I_{0-} is the amplitude of the left-going wave that results from termination of the line.