UNIVERSITY OF MARYLAND
Department of Physics
College Park, Maryland

Ph.D. PHYSICS QUALIFYING EXAMINATION - PART I

January 20, 2011 **9 a.m. - 1 p.m.**

Do any four problems.

Problems I. 1, I. 2, I. 4 and I. 5 are each worth 25 points.

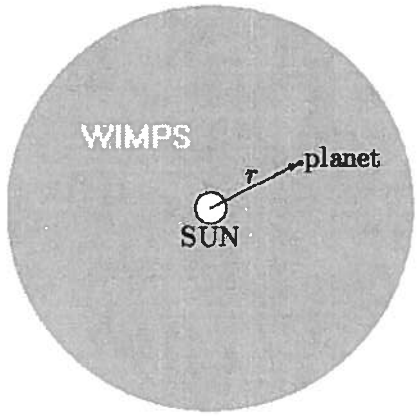
Problem I. 3 Statistical Mechanics is worth 40 points.

Put all answers on your answer sheets.

Be sure your Qualifier ID Number is at the top right corner of each sheet and turn in solutions to four problems only. If five solutions are turned in we will grade # 1 - # 4.

Problem I.1

A solar system is immersed in a uniform spherical cloud of Weakly-Interacting Massive Particles (WIMPS) of mass density ρ. The sun is at rest at the center of the cloud. A planet of mass m is located at a radius r from the sun. Assume that the planet and the Sun can be considered as point particles of mass m and $M_\odot$ respectively, with $M_\odot \gg m$.



(a) **[3 points]** Show that the force on the planet can be written as

$$\boldsymbol{F} = -m\left(\frac{k}{r^2} + br\right)\hat{\boldsymbol{r}},$$

where $\hat{\boldsymbol{r}}$ is a unit vector in the radial direction. Express the constants k and b in terms of $M_\odot$, ρ and G (Newton's gravitational constant).

(b) **[3 points]** Find the potential energy $V(r)$ associated with the conservative force $\boldsymbol{F}$.

(c) **[3 points]** What is the conservation law associated with the spherical symmetry of this system? Use it to show that the motion of the planet is confined to a two-dimensional plane.

Write the Lagrangian L of the planet, using the distance r and angle θ as generalized coordinates.

(d) **[3 points]** Define canonical momenta conjugate to r and θ and obtain the Hamiltonian of the planet.

(e) **[3 points]** Using the conservation law identified in (c), reduce the Hamiltonian to that for a particle moving in an effective one-dimensional potential energy $V_{\text{eff}}(r)$, and find $V_{\text{eff}}(r)$.

(f) **[3 points]** It is observed that the planet moves in a circular orbit with radius $r = r_0$. Find an algebraic expression that relates the radius r_0, the force constants k, b, and the angular momentum ℓ (do not solve for r_0).

Find the angular frequency $\dot{\theta}$ in terms of k, b and r_0.

I.1 (Continued)

(g) **[4 points]** Now consider a planet on a *nearly* circular orbit $r \approx r_0$. Its radial motion is an oscillation about the circular orbit. Find the frequency ω of this small-amplitude oscillation and compare with $\dot{\theta}$ found in (f), for small $b \ll k/r^3$.

Discuss the deviation from Kepler's first law[1] due to the presence of the WIMPS, and find the orbit's precession frequency.

(h) **[3 points]** What is the shape of the orbits for planets located at radial distances such that $r \gg (k/b)^{1/3}$?

[1]Kepler's first law states that the orbit of every planet is an ellipse with the Sun at one of the two foci.

Problem I.2

A dielectric sphere of radius a and dielectric constant ϵ is placed into an otherwise uniform electric field $\boldsymbol{E} = E_0\hat{\boldsymbol{z}}$ applied along the z-axis in vacuum.

(a) **[5 points]** State the equation and boundary conditions needed to determine the electric potential Φ in the presence of the sphere in the static electric field.

(b) **[6 points]** Using spherical coordinates, solve for the potential Φ inside and outside of the sphere.

(c) **[5 points]** Find the surface charge density σ_p on the sphere due to electric polarization and use it to determine the corresponding dipole moment p of the sphere induced by the applied electric field.

(d) **[5 points]** Now consider an electromagnetic plane wave of frequency ω electrically polarized in the z-direction and propagating in the x-direction. Assume that the frequency is low enough that the above-derived electrostatic result for the induced dipole moment can be used (i.e., the wavelength is much larger than the radius of the sphere). If the time-averaged Poynting flux in the incident plane wave is S, calculate the time-averaged Poynting flux due to scattering from the sphere in the y-direction at a distance r from the sphere.

(e) **[4 points]** The setting Sun appears to be red, but the sky is blue. Why?

Possibly useful information for parts (d) and (e):

Far from a z-oriented, harmonically oscillating electric dipole, $p\cos(\omega t)$, the electric field in vacuum expressed in (r, θ, ϕ) spherical coordinates is given in SI units by

$$E_\theta(r,\theta,\phi) = -p\sin\theta\left(\frac{\omega}{c}\right)^2 \frac{\cos[\omega(t - r/c)]}{4\pi\epsilon_0 r}.$$

Problem I.3

First consider a classical ideal gas at temperature T consisting of N molecules and initially confined in a volume V_i. Then the gas is allowed to expand to a final volume V_f in two different ways:

(a) **[8 points]** *Free expansion.* The gas is thermally insulated from its environment and experiences free irreversible expansion into a vacuum. Calculate the entropy change of the gas $\Delta S_{\text{irr}}^{\text{gas}} = S_f - S_i$ by comparing the number of accessible states before and after the expansion.

(b) **[8 points]** *Isothermal expansion.* The gas is in thermal contact with a reservoir of temperature T and experiences a slow reversible quasistatic expansion, e.g. produced by a slow motion of a piston that limits the gas volume. Calculate the work W done on the gas in this process, the change $\Delta U = U_f - U_i$ of the internal energy of the gas, and the heat Q transferred to the gas from the environment. Calculate the entropy change of the gas $\Delta S_{\text{rev}}^{\text{gas}} = S_f - S_i$ in this reversible process by using the formula $\Delta S = Q/T$. Compare your answers for $\Delta S_{\text{irr}}^{\text{gas}}$ and $\Delta S_{\text{rev}}^{\text{gas}}$. Are the two results the same or different? Explain why.

(c) **[5 points]** What are the entropy changes in the environment for these two cases: $\Delta S_{\text{irr}}^{\text{env}}$ and $\Delta S_{\text{rev}}^{\text{env}}$? What are the total entropy changes in the gas and the environment for these two cases: $\Delta S_{\text{irr}}^{\text{tot}} = \Delta S_{\text{irr}}^{\text{gas}} + \Delta S_{\text{irr}}^{\text{env}}$ and $\Delta S_{\text{rev}}^{\text{tot}} = \Delta S_{\text{rev}}^{\text{gas}} + \Delta S_{\text{rev}}^{\text{env}}$? Are $\Delta S_{\text{irr}}^{\text{tot}}$ and $\Delta S_{\text{rev}}^{\text{tot}}$ the same or different? Explain why.

(d) **[7 points]** Now consider a non-interacting degenerate Fermi gas made of N spin-1/2 fermions each of mass m and initially confined in a volume V_i at zero temperature $T = 0$. Calculate the Fermi momentum p_F, the Fermi energy E_F, and the energy per particle U/N. Express U/N in terms of E_F.

(e) **[6 points]** This Fermi gas is thermally insulated from its environment and experiences free irreversible expansion into a vacuum to a final volume V_f. Assume that V_f is sufficiently large so that the Fermi gas becomes non-degenerate, i.e. classical. Calculate the final temperature T_f of the gas after the expansion. Express T_f in terms of E_F obtained above.

(f) **[6 points]** Estimate by what factor V_f/V_i the initially degenerate Fermi gas needs to expand in order to become classical in the final state.

Hint: A gas behaves classically when the inter-particle separation $d = \sqrt[3]{N/V}$ is much greater than the thermal de Broglie length $\lambda = h/\sqrt{2\pi mkT}$, where h and k are the Planck and the Boltzmann constants.

Problem I.4

High-energy photons lose energy through three basic processes: ionization, Compton scattering, and pair production. In this problem, we consider the latter two processes.

(a) **[8 points]** First consider the problem of a photon Compton scattering off of a stationary electron. Derive the expression for the shift in wavelength of the photon $\Delta\lambda = \lambda' - \lambda$ as a function of the scattering angle ϕ of the photon.

(b) **[6 points]** In the remainder of the problem consider the pair production process. Determine the threshold condition on the energies E_1 and E_2 of two photons colliding at an angle θ, in order to produce an electron-positron pair e^+e^-.

(c) **[2 points]** Estimate the minimum energy that a high-energy photon, traveling through space, must have in order to produce an e^+e^- pair when it collides with a photon from the cosmic microwave background (CMB).

(d) **[2 points]** What effect would this have on the observability of ultra-high energy gamma-rays from distant sources?

(e) **[2 points]** Active Galactic Nuclei (AGN) consist of super-massive black holes located at the center of galaxies. Emanating from near the black hole and perpendicular to the plane of the galaxy are highly collimated relativistic matter outflows called jets. They emit large amounts of visible and ultraviolet light, with photon energies of order 10 eV in the jet's local rest frame. They also emit gamma rays. What is the threshold gamma ray energy E_2 for e^+e^- pair production between these gamma rays and the $E_1 = 10$ eV photons?

(f) **[5 points]** In apparent contradiction with your result in (d), gamma rays with energies as high as 10 TeV have been observed from these AGN jets when viewed from Earth along the jet axis. It has therefore been argued that the local region where these gamma rays are emitted must be moving with a high velocity v. Explain why this can resolve the contradiction and estimate the Lorentz factor $\gamma = \frac{1}{\sqrt{1-(v/c)^2}}$.

Possibly useful information:

- Peak photon energy of the CMB: 6×10^{-4} eV
- 1 TeV $= 10^{12}$ eV
- Mass of the electron: $m_e = 0.5\mathrm{MeV/c^2}$

Problem I.5

A transverse electromagnetic laser beam with electric field $E_{0y} \exp[i(kx - \omega t)]$ and magnetic field $B_{0z} \exp[i(kx - \omega t)]$ propagates in a plasma. In this problem we will focus on the electron motion and ignore the motion of the positive ions.

(a) **[6 points]** Write down Newton's equation of motion for an electron in the plasma and show that the laser electric field induces electron oscillation with velocity $\boldsymbol{v} = ie\boldsymbol{E}/m\omega$, where e is the electronic charge and m is the electron mass. Assuming that the electrons move independently, find an expression for the current density induced by the electric field for a plasma with electron density n.

(b) **[7 points]** Using Maxwell's equations with this induced current density, find the dispersion relation for electromagnetic waves $\omega(k)$, in terms of the plasma frequency $\omega_p = (ne^2/\epsilon_0 m)^{1/2}$ (in SI units) and the wavenumber k.

Note that for any vector field $\boldsymbol{A}$: $\nabla \times \nabla \times \boldsymbol{A} = \nabla(\nabla \cdot \boldsymbol{A}) - \nabla^2 \boldsymbol{A}$.

(c) **[6 points]** For $\omega \gg \omega_p$, find the group velocity with which laser beams will propagate through the plasma. Consider two laser pulses with center frequencies ω_1, $\omega_2 \gg \omega_p$ simultaneously incident on a plasma slab of length L: $0 < x < L$ from the left at $x = 0$ at time $t = 0$. What is the difference in time of exiting the plasma at $x = L$?

(d) **[6 points]** A laser beam with $\omega < \omega_p$ will be reflected by the plasma and the laser intensity will fall off exponentially inside the plasma. What is the exponential decay length scale (skin depth)?

Hint: Consider a wavenumber k with real and imaginary parts.