UNIVERSITY OF MARYLAND
Department of Physics
College Park, Maryland

Ph.D. PHYSICS QUALIFYING EXAMINATION - PART I

August 25, 2011 **9 a.m. - 1 p.m.**

Do any four problems.

Problems I. 1, I. 2, I. 4 and I. 5 are each worth 25 points.

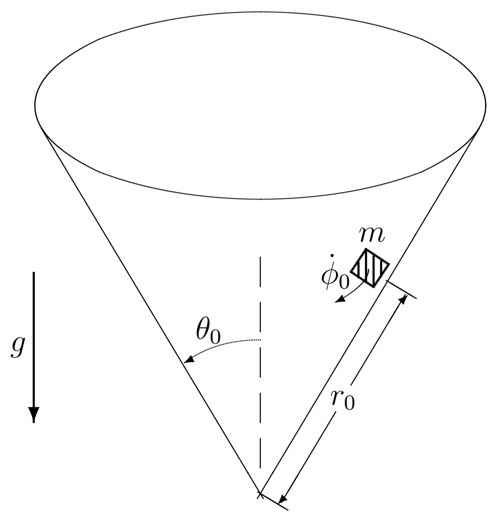
Problem I. 3 Statistical Mechanics is worth 40 points.

Put all answers on your answer sheets.

Be sure your Qualifier ID Number is at the top right corner of each sheet and turn in solutions to four problems only. If five solutions are turned in we will grade # 1 - # 4.

Problem I.1

Consider the frictionless motion of a mass m on the inner surface of a cone of half angle θ_0 that is oriented vertically in a uniform gravitational field g pointing downward along the symmetry axis of the cone. The mass is initially located at a distance r_0 from the vertex of the cone, and its initial velocity is azimuthal with an initial angular velocity $\dot{\phi}_0$. Assume that the mass m has negligible size and stays on the cone's surface throughout the motion.



(a) [**3 points**] What are the two constants of motion (c.o.m.) that can be used to describe the motion of the mass on the cone? Why do these constants exist?

(b) [**5 points**] Using the generalized coordinates r, the radial distance of the mass from the vertex of the cone, and ϕ, the azimuthal angle, write down the Lagrangian for the two-dimensional motion of the particle on the surface of the cone, and derive the equations of motion for the mass. Show that one c.o.m. follows immediately from these equations.

(c) [**5 points**] Another c.o.m. can be obtained by defining an effective potential $V(r)$ for the radial motion,

$$\dot{r}^2 + V(r) = 0, \tag{1}$$

where $V(r_0) = 0$. Derive $V(r)$ using the equations of motion.

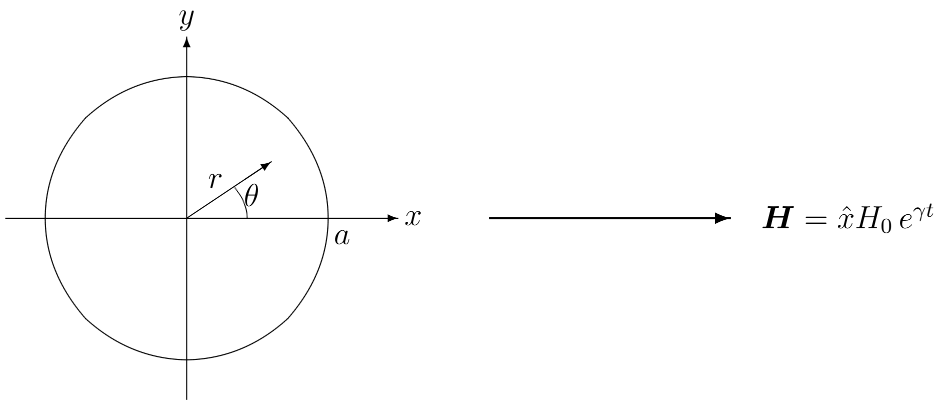
(d) [**3 points**] What are the limiting forms for the potential $V(r)$ as $r \to 0$ and $r \to \infty$? What do these limits imply about the radial motion of the mass?

(e) [**5 points**] Evaluate the slope of the potential $V' = dV/dr$ at the injection location r_0. Sketch the form of $V(r)$ for the three cases, $V'(r_0) < 0$, $V'(r_0) = 0$ and $V'(r_0) > 0$. Qualitatively discuss the motion of the mass for the three cases.

(f) [**4 points**] From $V(r)$ calculate the minimum point r_s of the potential. Assume that the excursion of the mass from r_s is small, expand the potential around this location, and calculate the bounce frequency ω_b of the mass in the potential well.

Problem I.2

Consider a long cylindrical rod of radius a and uniform electrical conductivity σ with its axis (at $r = 0$) parallel to $\hat{z}$ (see Figure). An external uniform time-dependent magnetic field $\boldsymbol{H} = \hat{x} H_0 \, e^{\gamma t}$ is applied along the x direction, with $\gamma > 0$. In this problem, we study how the magnetic field is modified due to the presence of the conducting rod.



(a) [**4 points**] First consider the uniform time-dependent magnetic field as stated above in free space without a rod. Many different vector potentials can describe this magnetic field. Show that one possible choice appropriate for this geometry is $\boldsymbol{A} = \hat{z} A_z$ (i.e. $A_x = A_y = 0$) and find A_z.

Having found A_z, rewrite it as a function of the cylindrical coordinates r and θ (see the Figure) for future use, i.e. as $A_z(r, \theta, t)$.

What is the electric field produced by this vector potential in the case where the electric potential is zero ($A_0 = 0$)?

(b) [**4 points**] Now let us introduce the conducting rod. Electric currents are induced in the rod and modify the magnetic field. In what direction do the currents flow? What partial differential equations does A_z satisfy inside and outside the rod? Neglect the displacement current, i.e. neglect the term with the second derivative in time $\partial^2 A_z / \partial t^2$.

(c) [**5 points**] Show that the solution for A_z inside the rod has the form $A_z = C_0 \, e^{\gamma t} \sin\theta \, I_1(\rho)$, where $I_1(\rho)$ is the modified Bessel function, and $\rho = r/\ell$. Find a value for the parameter ℓ and explain its physical meaning. Useful information about the modified Bessel function is given at the end of the problem.

(d) [**4 points**] Show that possible solutions for A_z outside the rod have the form $A_z \propto e^{\gamma t} \sin\theta \, r^{\alpha}$ and find values of α. There are two possible values of α, so a general solution outside the rod can be written as $A_z = e^{\gamma t} \sin\theta \, (C_1 r^{\alpha_1} + C_2 r^{\alpha_2})$.

(e) [**4 points**] Formulate continuity conditions for the vector potential A_z and its spatial derivatives at the boundary between the rod and vacuum at $r = a$. Using these conditions and the boundary condition that A_z approaches to the unperturbed solution found in Part (a) at $r \to \infty$, determine the coefficients C_0, C_1, and C_2 in terms of H_0, a, ℓ, and the values $I_1(a/\ell)$ and $I_1'(a/\ell)$ of the modified Bessel function and its derivative at $\rho = a/\ell$.

I.2 (Continued)

(f) [**4 points**] Sketch the magnetic field lines outside the rod for the case where the conductivity of the rod is very large, $\sigma \gg 1/\gamma\mu_0 a^2$. Compare the values of ℓ and a in this case.

Possibly useful information:

The Laplacian operator in two dimensions in cylindrical coordinates is

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}.$$

The modified Bessel function $I_1(\rho)$ satisfies the equation

$$\frac{1}{\rho}\frac{d}{d\rho}\rho\frac{d}{d\rho}I_1(\rho) - \left(1 + \frac{1}{\rho^2}\right) I_1(\rho) = 0, \qquad I_1(0) = 0.$$

$I_1(\rho)$ is a monotonically increasing function of ρ, and $I_1(\rho) \approx e^{\rho}/\sqrt{2\pi\rho}$ for $\rho \gg 1$.

Problem I.3

Consider a gas consisting of N indistinguishable spinless non-relativistic particles of mass m contained in a volume V. The temperature of the system is T.

(a) [**8 points**] Under what condition (formulated in terms of V, N, T, m, and fundamental constants) can the translational motion of the particles be treated using classical statistical mechanics, as opposed to quantum statistics?

(b) [**8 points**] Suppose the particles interact with a hard-core repulsive potential

$$U_{\text{int}}(r) = \begin{cases} \infty, & r < a, \\ 0, & r > a, \end{cases}$$

where r is the distance between two particles, and $a > 0$ is the range of interaction.

Assuming that the translational motion of the particles can be treated classically, calculate the partition function Z_{tr} of the system, up to an overall multiplicative constant.

Using Z_{tr}, calculate the free energy F of the system and the pressure P (see useful equations at the bottom). Compare the obtained equation of state with that of an ideal gas and check the limiting case $a \to 0$.

(c) [**8 points**] Now suppose that the particles in the gas have some internal structure and can vibrate by expanding and contracting (they have a "breathing mode"). These oscillations are approximately harmonic and have the frequency ω.

Calculate the partition function of the system $Z = Z_{\text{tr}}\, Z_{\text{vib}}$ taking into account both translational and vibrational motion. Assume that the translational motion can be treated classically, but the vibrations have to be treated fully quantum-mechanically. (You can use Z_{tr} from the previous part.)

(d) [**8 points**] Using Z, calculate the internal energy E of the system and the heat capacity C_V at constant volume (see useful equations below).

(e) [**8 points**] Under what condition can the vibrational motion be treated using classical statistical mechanics, as opposed to quantum statistics? Find the limiting expressions for E and C_V in this case and verify that they agree with the equipartition theorem.

Possibly useful equations:

$$F = -k_B T \ln Z, \quad \text{where } k_B \text{ is the Boltzmann constant,} \qquad P = -\left(\frac{\partial F}{\partial V}\right)_{T,N}$$

$$E = -\left(\frac{\partial \ln Z}{\partial \beta}\right)_{V,N}, \quad \text{where} \quad \beta = \frac{1}{k_B T}, \qquad C_V = \left(\frac{\partial E}{\partial T}\right)_{V,N}$$

$$\int_{-\infty}^{\infty} dx\, e^{-\alpha x^2} = \sqrt{\frac{\pi}{\alpha}}$$

$$\sum_{n=0}^{\infty} b^n = \frac{1}{1-b}, \quad b < 1$$

Problem I.4

When cosmic ray protons impact atomic nuclei in the upper atmosphere, pions are created. These pions decay within a relatively short distance into muons and neutrinos:

$$\pi^{\pm} \rightarrow \mu^{\pm} + \nu.$$

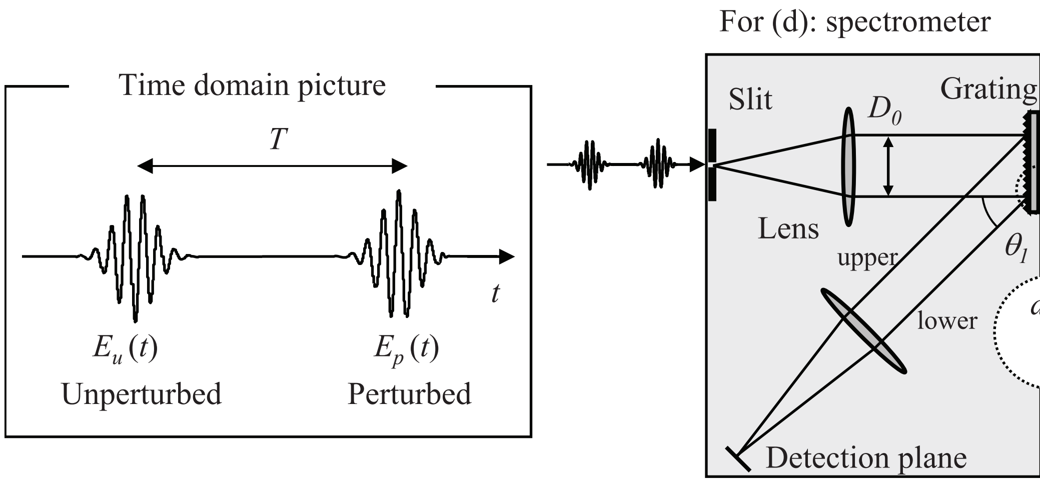
(a) [**12 points**] Using the conservation of energy and momentum, find expressions for the energy and momentum for the muon in the rest frame of the pion in terms of the pion and muon rest masses, M and m respectively. Assume that the neutrino is massless.

(b) [**5 points**] In the Earth's frame these muons are created with a typical total energy of 6 GeV. The muon has a rest mass of $m = 105.7$ MeV and a lifetime of $\tau = 2.2 \times 10^{-6}$ s. What is the muon's lifetime in the Earth's rest frame?

(c) [**8 points**] Muons are created at an altitude of 15 km above sea level. What fraction of the muons moving downward would reach sea level? Ignore further interaction of the muons with the atmosphere.

Problem I.5

Consider two identical optical pulses separated by time T (see the left panel in the figure below). Assume an unknown modulation, $Ae^{i\phi}$ (amplitude and phase), is applied to only one of the pulses. In this case, the unperturbed and perturbed pulse fields can be expressed as $E_u(t) = E_0(t)e^{-i\omega_0 t}$ and $E_p(t) = Ae^{i\phi}E_u(t-T)$, respectively. Here, we are interested in determining $Ae^{i\phi}$ by observing the interference pattern that the two pulses make inside a spectrometer. The argument given here forms the basis for frequency-domain (or spectral) interferometry.



(a) [**10 points**] Derive an expression for the total spectral intensity $I(\omega) = |E_u(\omega) + E_p(\omega)|^2$ that the two-pulse system produces, in terms of $\tilde{E}_u(\omega)$, T, A, ϕ, and ω, where $\tilde{E}_u(\omega) = \int_{-\infty}^{+\infty} E_u(t)e^{i\omega t}dt$ is the spectral amplitude of the unperturbed pulse.

(b) [**5 points**] Sketch $I(\omega)$ for $A = 1$, $\phi = \pi$, and $\omega_0 T = 100 \times (2\pi)$, assuming that the two pulses are Gaussian-shaped as $E_0(t) = E_0 e^{-t^2/\tau^2}$.

(c) [**5 points**] Assume that the modulation $Ae^{i\phi}$ originates from the complex index of refraction, $N = n + i\kappa$, of the material in which only the perturbed pulse propagates over a distance l. Derive an expression for n and κ, respectively, in terms of A, ϕ, ω_0, l, and c, where c is the speed of light in vacuum.

(d) [**5 points**] Consider the two pulses entering a grating-based spectrometer (see the right panel in the figure above). Here, the pulses, having transverse size D_0, are normally incident upon the grating with groove spacing d. Find a physical condition for the two pulses to interfere at the detection plane of the spectrometer. Consider only the first-order diffraction.

Hint: Calculate the difference in propagation times from the entrance slit to the detection plane between the upper and lower parts of the diffracted beam. Then consider under what condition the two pulses can overlap in time on the detection plane of the spectrometer.