UNIVERSITY OF MARYLAND
Department of Physics
College Park, Maryland

Ph.D. PHYSICS QUALIFYING EXAMINATION - PART I

August 23, 2012 **9 a.m. - 1 p.m.**

Do any four problems.

Problems I. 1, I. 2, I. 4 and I. 5 are each worth 25 points.

Problem I. 3 Statistical Mechanics is worth 40 points.

Put all answers on your answer sheets.

Be sure your Qualifier ID Number is at the top right corner of each sheet and turn in solutions to four problems only. If five solutions are turned in we will grade # 1 - # 4.

Problem I.1

In this problem, we study torsion and torsional waves in a uniform cylindrical rod of length ℓ and radius R with a constant mass density ρ, a shear modulus G, and an area moment of inertia J. The latter is defined as $J = \int_0^R r^2\, d^2r$, where $\boldsymbol{r} = (y, z)$ is the transverse coordinate vector, and is related to the usual mass moment of inertia I by $I = \rho J \ell$. The twist angle of the rod, $\phi(x, t)$, is a function of position x along the rod and of time t.

(a) [**3 points**] With one end of the rod ($x = 0$) held fixed, a twisting torque T is applied to the other end ($x = \ell$). In the case where the applied torque and twist angle are time independent, the local and total twist angles are,

$$\phi(x) = \frac{Tx}{GJ}, \qquad \Delta\phi = \phi(\ell) - \phi(0) = \frac{T\ell}{GJ}.$$

How much work was done to twist the rod? Express your answer in terms of $\Delta\phi$ and other parameters, eliminating T from the answer.

(b) [**3 points**] Now consider the case of a general dependence $\phi(x, t)$. Applying the result of Part (a) to a small section dx of the rod length and using the work-energy relation, show that the potential energy of the rod is

$$V = \frac{GJ}{2} \int_0^\ell \left(\frac{\partial \phi}{\partial x}\right)^2 dx.$$

(c) [**4 points**] Write down an expression for the kinetic energy K of the rod in terms of $\phi(x, t)$, J, ρ, and an integral along the rod. What is the system Lagrangian, L?

(d) [**3 points**] If both ends of the rod are free (i.e., not clamped or constrained in any way), show that $\partial\phi/\partial x = 0$ at $x = 0$ and $x = \ell$.

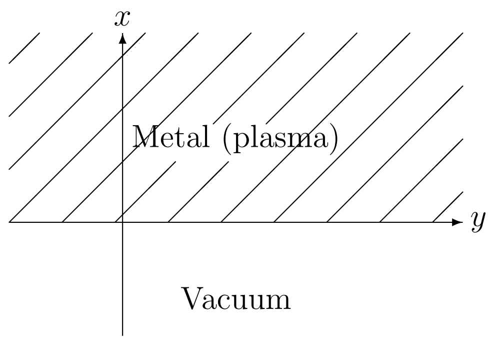
(e) [**7 points**] Using the Lagrangian L and Hamilton's principle, find a partial differential equation for $\phi(x, t)$ describing torsional motion of the rod. Assuming that both ends of the rod are free, indicate how the boundary conditions appear in the derivation.

(f) [**5 points**] What is the speed of propagation of torsional waves along the rod? What are the allowed frequencies of the small-amplitude torsional oscillations of the rod, if both ends of the rod are free?

Problem I.2

In microwave scattering on a metal, a resonant absorption was observed at a frequency lower than the metal's plasma frequency $\omega_p = \sqrt{4\pi n e^2/m}$, where e is electron charge, m electron effective mass, and n number density of free electrons. This led to an important area of applications called "plasmonics". Here we study a wave propagating along the interface between a dielectric and a metal and derive the frequency of this surface plasma wave.

Let the metal (treated as a plasma with immobile ions and cold electrons of density n) occupy the half space $x > 0$. Assume that the space at $x \le 0$ is occupied by vacuum (with dielectric constant $\epsilon = 1$).



(a) [**5 points**] Suppose that an electric field $\boldsymbol{E}(t) = \boldsymbol{E}_0 e^{-i\omega t}$ exists in the plasma. What is the motion of the free electrons due to the force from this field, according to Newton's law? Express the electron's displacement, $\boldsymbol{r}$, in terms of the field $\boldsymbol{E}$.

(b) [**5 points**] Find the polarization $\boldsymbol{P}$ (electron dipole moment density) induced by this electric field. Then use $\boldsymbol{D} = \boldsymbol{E} + 4\pi\boldsymbol{P} = \epsilon(\omega)\boldsymbol{E}(\omega)$ to show that the dielectric function of the plasma is $\epsilon(\omega) = 1 - (\omega_p/\omega)^2$.

(c) [**5 points**] If $\epsilon < 0$ in the metal (which is the case for $\omega < \omega_p$), there exist solutions of Maxwell's equations that are concentrated near the surface and propagate along the surface. For a wave with wave vector $\boldsymbol{k} = k\boldsymbol{j}$, propagating in the y-direction, and in the quasistatic approximation $\omega \ll ck$, the electric field can be derived from a potential,

$$\boldsymbol{E} = -\boldsymbol{\nabla}\Phi \qquad \Phi(x, y, t) = \phi(x, y)e^{-i\omega t}, \tag{1}$$

where ϕ satisfies the Poisson equation with sources only on the surface. Show that the equation $\boldsymbol{\nabla}^2\phi = 0$, valid for both $x > 0$ and $x < 0$, has a solution of the form

$$\phi(x, y) = \begin{cases} Ae^{-\lambda x + iky} & \text{for } x > 0 \\ Be^{\lambda x + iky} & \text{for } x < 0 \end{cases} \tag{2}$$

and find a relation between λ and k.

(d) [**5 points**] From Eqs. (1) and (2), derive the electric field $\boldsymbol{E}$ for $x > 0$ and $x < 0$.

From the boundary conditions at $x = 0$ (continuity of the tangential component of $\boldsymbol{E}$ and normal component of $\boldsymbol{D}$), derive a relation between the coefficients A and B and obtain the frequency ω of the surface plasma wave. Does this ω depend on k?

(e) [**5 points**] In a diagram like the figure above, sketch the electric surface charges (by $+$ and $-$) and the electric field lines of the surface plasma waves at $t = 0$.

Problem I.3

In this problem, we discuss the physics of a gas of electrons, positrons, and photons in thermal equilibrium.

(a) **[5 points]** To begin, consider a gas of non-interacting spin-1/2 fermions with a dispersion relation

$$E(\boldsymbol{k}) = \sqrt{m^2c^4 + \hbar^2c^2k^2}$$

confined to a box of volume V. Let F denote the free energy $F = -k_BT \ln Z$, where

$$Z = \sum_{\text{states}} e^{-(E-\mu N)/k_BT}$$

is the grand partition function. Here, the sum runs over all many-particle states, E is the total energy of the state, N is the total number of particles in the state, μ is the chemical potential, and T is the temperature. Write down an integral expression for the free energy F of the fermion gas of the form

$$F_{\text{fer}}(V,T,\mu) = V \int d^3k \; g(k) \tag{1}$$

and find the function g.

(b) **[5 points]** In the same way, write down an integral expression for the free energy F of a photon gas, confined in a box of volume V:

$$F_{\text{ph}}(V,T) = V \int d^3k \; h(k) \tag{2}$$

and find the function h. What is the chemical potential of photons?

(c) **[5 points]** We are now ready to consider a gas of electrons, positrons, and photons. Throughout the problem, we will assume the gas has equal numbers of electrons and positrons. However, the total number of these particles can change via the pair production process $\gamma + \gamma \leftrightarrow e^- + e^+$. Show that $\mu_{e^-} = \mu_{e^+} = 0$ in this system in equilibrium.

(d) **[5 points]** Changing the integral in Eq. (2) to a dimensionless variable, show that the photon free energy has the form

$$F_{\text{ph}}(V,T) = V\,(k_BT)^a\,B_{\text{ph}} \tag{3}$$

and obtain the exponent a. Then determine the coefficient B_{ph} using the integral

$$\int_0^\infty \frac{x^3}{e^x - 1}\,dx = \frac{\pi^4}{15}. \tag{4}$$

Hint: Write the integral (2) in spherical coordinates as $\int_0^\infty h(k)\,4\pi\,d(k^3/3)$ and integrate by parts.

I.3 (Continued)

(e) [**5 points**] Consider the integral in Eq. (1) for $\mu = 0$ in the limit of high temperatures $k_B T \gg mc^2$. Show that, in this limit, the fermion free energy has the form

$$F_{\text{fer}}(V,T) = V\,T^b\,B_{\text{fer}} \tag{5}$$

and obtain the exponent b. Then determine the coefficient B_{fer} using the integral

$$\int_0^\infty \frac{x^3}{e^x+1}\,dx = \frac{7\pi^4}{120}. \tag{6}$$

Hint: Write the integral (1) in spherical coordinates as $\int_0^\infty g(k)\,4\pi\,d(k^3/3)$ and integrate by parts.

(f) [**5 points**] Using the results of Parts (d) and (e), find the total free energy F, entropy S, and energy U of the electron, positron, and photon mixture in the high-temperature limit $k_B T \gg mc^2$.

(g) [**5 points**] Now consider the low-temperature limit $k_B T \ll mc^2$. Show that the free energy, entropy, and energy of the electron-positron gas are exponentially small in this limit.

(h) [**5 points**] Imagine that we slowly expand the box from an initial volume V_i to a final volume V_f, keeping it thermally insulated during the process. Assume that the initial temperature satisfies the condition $k_B T_i \gg mc^2$, while the final temperature satisfies the condition $k_B T_f \ll mc^2$. Find T_f in terms of V_f, V_i, and T_i. You can neglect the electrons and positrons in the final system, given the result of Part (g).

Problem I.4

A rocket accelerates by ejecting part of its mass (the fuel) as exhaust. The matter of the exhaust is ejected at a constant velocity u with respect to the rocket. The rocket starts at rest with mass M_0 and reaches a terminal speed V at burnout, when all of its fuel is used up. Its rest mass at burnout is M. You are to find the ratio M/M_0 in terms of V and u.

(a) [**6 points**] First solve the problem non-relativistically, when the rocket's speed $v \ll c$.

(b) [**6 points**] In the relativistic case, we may set $c = 1$ and consider the acceleration as an addition of many small velocity increases. Let the rocket's momentary rest mass be m, and let a small amount of rest mass dm be ejected in the rocket's momentary rest frame. Since the resulting velocity increase dv is small, a result analogous to Part (a) applies. Hence write dv in terms of m, dm and u.

(c) [**7 points**] Add these velocity increases relativistically to get the terminal speed V and solve for M/M_0.

Hint: Relativistic addition of velocity is an ordinary addition of the rapidity (hyperbolic angle) $\alpha = \tanh^{-1} v$. By expressing dv in terms of $d\alpha$, your equation in Part (b) can be integrated directly.

(d) [**6 points**] Find the ratio M/M_0 for a photon rocket, which emits radiation at the speed $c = 1$, for example by combining matter and antimatter in a controlled way to yield high-energy gamma rays.

Problem I.5

In this problem, we study propagation of plane waves in a homogeneous, nonpermeable ($\mu = 1$), but anisotropic dielectric medium, which is characterized by a symmetric dielectric tensor ϵ_{ij} such that $D_i = \sum_{j=x,y,z} \epsilon_{ij} E_j$.

(a) [**5 points**] Starting from Maxwell's equations, show that a plane wave solution

$$\boldsymbol{E}(\boldsymbol{r}, t) = \tilde{\boldsymbol{E}}(\boldsymbol{k}, \omega)\, e^{-i\omega t + i\boldsymbol{k}\cdot\boldsymbol{r}} \tag{1}$$

with the frequency ω and wave vector $\boldsymbol{k}$ must satisfy the following equation in the Gaussian system

$$(\boldsymbol{k} \cdot \tilde{\boldsymbol{E}})\, \boldsymbol{k} - k^2 \tilde{\boldsymbol{E}} + \frac{\omega^2}{c^2} \boldsymbol{\epsilon} \cdot \tilde{\boldsymbol{E}} = 0, \tag{2}$$

where c is the speed of light, and $(\boldsymbol{\epsilon} \cdot \tilde{\boldsymbol{E}})_i \equiv \sum_j \epsilon_{ij} \tilde{E}_j$.

(b) [**10 points**] Suppose x, y, and z are the directions that diagonalize the tensor

$$\epsilon_{ij} = \begin{pmatrix} \epsilon_{xx} & 0 & 0 \\ 0 & \epsilon_{yy} & 0 \\ 0 & 0 & \epsilon_{zz} \end{pmatrix}. \tag{3}$$

Consider a linearly polarized plane wave (1) of the frequency ω traveling in this medium along the direction $\hat{z}$, so that $\boldsymbol{k} \| \hat{z}$. From Eq. (2), find the two possible wave numbers $k_{1,2}$ for this wave and their respective polarizations $\tilde{\boldsymbol{E}}_{1,2}$, as well as the corresponding wave lengths $\lambda_{1,2}$.

(c) [**10 points**] Suppose a plane wave of the frequency ω propagates along the direction $\hat{z}$ and is polarized along the direction $\tilde{\boldsymbol{E}} \| (\hat{x} + \hat{y})$ at $z = 0$. At what distance L does the polarization $\tilde{\boldsymbol{E}}$ of the wave turn 90° to become $\tilde{\boldsymbol{E}} \| (\hat{x} - \hat{y})$?

Additional information. For any vector $\boldsymbol{V}$,

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{V}) = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{V}) - \boldsymbol{\nabla}^2 \boldsymbol{V}.$$

Maxwell's equations in the absence of free charges and currents are

$$\boldsymbol{\nabla} \cdot \boldsymbol{D} = 0,$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0,$$

$$\boldsymbol{\nabla} \times \boldsymbol{E} = -\frac{1}{c} \frac{\partial \boldsymbol{B}}{\partial t},$$

$$\boldsymbol{\nabla} \times \boldsymbol{H} = \frac{1}{c} \frac{\partial \boldsymbol{D}}{\partial t}.$$