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Ph.D. PHYSICS QUALIFYING EXAMINATION - PART II

August 24, 2012 **9 a.m. - 1 p.m.**

Do any four problems. Each problem is worth 25 points.

Put all answers on your answer sheets.

Be sure your Qualifier ID Number is at the top right corner of each sheet and turn in solutions to four problems only. If five solutions are turned in we will grade # 1 - # 4.

Problem II.1

Last year, there was experimental work on spectroscopy of anti-hydrogen (a bound state of an antiproton and a positron). In this problem, we study the ground-state energy level splitting in either hydrogen or anti-hydrogen caused by the hyperfine structure and the Zeeman effect.

The electron in the ground state $\Psi_{1,0,0}$ of the hydrogen atom experiences an effective magnetic field

$$\boldsymbol{B}_p = \frac{2}{3}\,\mu_0\, g_p\, \mu_p\, \frac{\boldsymbol{S}_p}{\hbar}\,|\Psi_{1,0,0}(0)|^2$$

produced by the magnetic moment $g_p\mu_p\boldsymbol{S}_p$ of the proton, where $\boldsymbol{S}_p$, $\mu_p = e\hbar/2m_p$, and g_p are the proton spin, magneton, and g-factor. The electron probability density at the origin is $|\Psi_{1,0,0}(0)|^2 = 1/\pi a_0^3$, where a_0 is the Bohr radius. The electron magnetic moment is $\boldsymbol{\mu}_e = -g_e\,\mu_B\,\boldsymbol{S}_e/\hbar$, where $\boldsymbol{S}_e$, $\mu_B = e\hbar/2m_e$, and g_e are the electron spin, magneton and g-factor. Numerical values of the parameters are given at the end of the problem.

(a) **[3 points]** The electron magnetic moment interacts with the proton magnetic field via the Hamiltonian $H_{ep} = -\boldsymbol{\mu}_e \cdot \boldsymbol{B}_p$. Show that this Hamiltonian can be written in the form $H_{ep} = A\,(\boldsymbol{S}_p \cdot \boldsymbol{S}_e)/\hbar^2$ and obtain a formula for the coefficient A. Then substitute the numbers and calculate the numerical value of A in Joules.

(b) **[6 points]** The total spin $\boldsymbol{S} = \boldsymbol{S}_p + \boldsymbol{S}_e$ of the hydrogen atom in the ground state is the vector sum of the nuclear spin $\boldsymbol{S}_p$ and the electron spin $\boldsymbol{S}_e$. Rewrite the Hamiltonian $H_{ep} = A\,(\boldsymbol{S}_p \cdot \boldsymbol{S}_e)/\hbar^2$ in terms of the eigenvalues of the operators $\boldsymbol{S}_p^2$, $\boldsymbol{S}_e^2$, and $\boldsymbol{S}^2$. What are the allowed eigenvalues of these operators? What are the energy levels of H_{ep} in the states with $S = 1$ and $S = 0$? Express the hyperfine energy splitting ΔE_{hf} between the states with $S = 1$ and $S = 0$ in terms of A and calculate the corresponding frequency $f = \Delta E_{\text{hf}}/h$ in THz.

(c) **[8 points]** Now consider the effect of an applied external magnetic field B. The Hamiltonian of Zeeman interaction $H_Z = -\boldsymbol{\mu}_e \cdot \boldsymbol{B}$ couples the electron magnetic moment to the external field. (Neglect interaction of the external magnetic field with the proton magnetic moment, because $\mu_p \ll \mu_e$.) Calculate the change in energy levels of the hydrogen atom in the ground state $n = 1$ as a function of the applied magnetic field B by taking into account both the Zeeman effect and the hyperfine interaction. What happens when the external field B is comparable to B_p?

(d) **[6 points]** Plot your resulting energy levels versus the external field B. Label them according to the quantum numbers S and S^z, using the $B = 0$ point to identify the states. Comment on which of the states are strongly (linearly) dependent on the magnetic field B for small field.

(e) **[2 points]** Suppose you have a magnetic trap, where the magnetic field is zero at the center and increases in magnitude in all directions away from the origin. Which of quantum states discussed in Part (d) can be trapped in this device?

Numerical values in the SI units of N, A, J, and T for Newton, Ampere, Joule, and Tesla: $\mu_0 = 4\pi \times 10^{-7}$ N A^{-2}, $\mu_B = 9.27 \times 10^{-24}$ J T^{-1}, $g_e = 2$, $\mu_p = 5.0510 \times 10^{-27}$ J T^{-1}, $g_p = 5.58$, $a_0 = 5.29 \times 10^{-11}$ m, $h = 6.626 \times 10^{-34}$ J s.

Problem II.2

An atom has two internal energy eigenstates $|g\rangle$ and $|e\rangle$ with the energy difference $E_e - E_g = \hbar\omega_0 > 0$. The two states are connected by the atomic electric dipole operator $\hat{\boldsymbol{\mu}} = \boldsymbol{\mu}_0(\hat{\sigma}_+ + \hat{\sigma}_-)$, where $\hat{\sigma}_+ = |e\rangle\langle g|$ and $\hat{\sigma}_- = |g\rangle\langle e|$ are the raising and lowering operators of the two-level system. The atom has mass m and is confined in a 1D harmonic potential with the angular frequency of motion ω_m. The energy eigenstates of the motion of the atom in the harmonic potential are denoted as $|n\rangle$, where n is an integer, $n \geq 0$. These states are connected by the raising and lowering operators: $\hat{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$ and $\hat{a}|n\rangle = \sqrt{n}|n-1\rangle$.

An electromagnetic wave propagating along the direction of harmonic motion with frequency ω and wavevector $k = \omega/c$ exerts the electric field $\boldsymbol{E}(\hat{x}, t) = \boldsymbol{E}_0 \cos(k\hat{x} - \omega t)$ on the atom. Here the operator of the atomic coordinate $\hat{x}$ can be written in terms of the raising and lowering operators of the oscillator: $\hat{x} = x_0(\hat{a}^\dagger + \hat{a})$, where $x_0 = \sqrt{\hbar/2m\omega_m}$.

The Hamiltonian $\hat{H} = \hat{H}_0 + \hat{V}$ of the entire system consists of the unperturbed Hamiltonian

$$\hat{H}_0 = -\frac{1}{2}\hbar\omega_0\,\hat{\sigma}_z + \hbar\omega_m \hat{a}^\dagger\hat{a}, \tag{1}$$

where $\hat{\sigma}_z$ is the Pauli matrix operating on the states $|g\rangle$ and $|e\rangle$, and the perturbation

$$\hat{V} = -\hat{\boldsymbol{\mu}} \cdot \boldsymbol{E}(\hat{x}, t) = -2\hbar\Omega\,(\hat{\sigma}_+ + \hat{\sigma}_-)\cos(k\hat{x} - \omega t), \qquad \Omega = (\boldsymbol{\mu}_0 \cdot \boldsymbol{E}_0)/2\hbar. \tag{2}$$

(a) **[2 points]** Suppose $|\psi(t)\rangle$ represents a time-dependent wavefunction of the system, which satisfies the Schrödinger equation $i\hbar\partial|\psi(t)\rangle/\partial t = \hat{H}(t)|\psi(t)\rangle$. Let us introduce a modified wavefunction $|\tilde{\psi}(t)\rangle$ defined via the following unitary transformation $|\psi(t)\rangle = e^{-i\hat{H}_0 t/\hbar}|\tilde{\psi}(t)\rangle$. Show that the modified wave function satisfies a modified Schrödinger equation $i\hbar\partial|\tilde{\psi}(t)\rangle/\partial t = \tilde{V}(t)|\tilde{\psi}(t)\rangle$, where $\tilde{V}(t) = e^{i\hat{H}_0 t/\hbar}\,\hat{V}(t)\,e^{-i\hat{H}_0 t/\hbar}$. This representation of the wavefunction is called the "interaction representation".

(b) **[7 points]** Using Eqs. (1) and (2), calculate the perturbation $\tilde{V}(t)$ in the interaction representation.

Hint: First calculate the operators $e^{i\hat{H}_0 t/\hbar}\,\hat{\sigma}_\pm\,e^{-i\hat{H}_0 t/\hbar}$ and $e^{i\hat{H}_0 t/\hbar}\,\hat{a}^{(\dagger)}\,e^{-i\hat{H}_0 t/\hbar}$ in the interaction representation and then substitute the results into Eq. (2). Keep in mind that the operators $\hat{\sigma}_\pm$ and $\hat{a}^{(\dagger)}$ connect states with different energies.

(c) **[4 points]** Now suppose that the detuning $\delta = \omega - \omega_0$ is small compared with Ω, i.e., $|\delta| \ll \Omega$. In the formula for $\tilde{V}(t)$ obtained in Part (b), neglect the terms that oscillate fast as $e^{\pm i(\omega+\omega_0)t}$ and retain only the terms that oscillate slowly as $e^{\pm i\delta t}$. This approximation is called the "rotating wave approximation".

(d) **[7 points]** Now suppose the detuning is an integer multiple of the harmonic oscillator frequency: $\delta = j\omega_m$, where j is an integer (positive or negative). Assuming that the parameter $kx_0 \ll 1$ is small, perform the Taylor expansion of $\tilde{V}(t)$ obtained in Part (c) in powers of kx_0 (which is known as the "Lamb-Dicke expansion"). Keep only the lowest-order stationary term in the expansion (i.e., the term that does not depend on time t) and neglect all other terms. Show that this term connects the quantum states $|g\rangle|n\rangle$ and $|e\rangle|n+j\rangle$ and calculate the matrix element $V(n,j) = \langle e|\langle n+j|\,\tilde{V}\,|g\rangle|n\rangle$ for

II.2 (Continued)

$j \geq 0$ and $j < 0$. Compare the photon energy $\hbar\omega$ with the change of the total energy of the atom and verify that your obtained result is consistent with energy conservation.

(e) **[5 points]** Suppose the system is initially prepared in the state $|g\rangle|n\rangle$ at $t = 0$. Taking into account the matrix element $V(n, j)$ derived in Part (d), show that the system will be in a superposition of the states $|g\rangle|n\rangle$ and $|e\rangle|n + j\rangle$ at a subsequent time. Calculate the probabilities of finding the system in each of these two states at a time t. Show that these probabilities oscillate in time and calculate the frequency and the period of these oscillations. Sketch the probabilities as a function of t and calculate the time to make a complete transition (with 100% probability) from the state $|g\rangle|n\rangle$ to the state $|e\rangle|n + j\rangle$. What is the name of these oscillations?

Problem II.3

For scattering off a spherically symmetric potential located at the origin, the wavefunction of a particle with the energy $E = \hbar^2 k^2/2m$ for large r is often written as

$$\psi(z, r, \theta) = e^{ikz} + f(\theta)\frac{e^{ikr}}{r}. \tag{1}$$

(a) [**2 points**] Interpret the two terms in Eq. (1).

(b) [**4 points**] By computing the incoming and outgoing probability fluxes, show that the differential cross-section is

$$\frac{d\sigma}{d\Omega} = |f(\theta)|^2.$$

In terms of a partial wave expansion,

$$f(\theta) = \frac{1}{k}\sum_{\ell=0}^{\infty}(2\ell+1)\, e^{i\delta_\ell}\, \sin\delta_\ell\, P_\ell(\cos\theta), \tag{2}$$

where δ_ℓ is the phase shift of the ℓ-th partial wave due to the potential, and P_ℓ is the Legendre polynomial. Below we will only study s-wave scattering with $\ell = 0$, which dominates when the energy E of the scattered particle is low, i.e., k is small.

(c) [**4 points**] Keeping only the term with $\ell = 0$ in Eq. (2), write down expressions for the differential and total cross-sections of scattering in terms of the phase shift δ_0. Verify that your result satisfies the optical theorem $\sigma = (4\pi/k)\mathrm{Im} f(0)$. What value of δ_0 maximizes the scattering cross-section?

(d) [**5 points**] Now consider a spherically-symmetric square potential well of depth V and radius R. For this attractive potential, derive a general transcendental equation relating the s-wave phase shift δ_0 to k, V, and R. Do not assume that k is small here.

Hint: Introduce the radial wave function $u(r)$ via the relation $\psi(r) = u(r)/r$, apply the appropriate boundary condition at $r = 0$, and match the boundary conditions at $r = R$. At $r \to \infty$, the scattered wave in the ℓ-th channel goes as $u_\ell(r) \propto \sin(kr - \ell\pi/2 + \delta_\ell)$.

(e) [**5 points**] Now consider the case of small k, where $kR \ll 1$ and $\hbar^2 k^2/2m \ll V$. Using the general equation derived in Part (d), find the values of V that give $|\delta_0| = \pi/2$ (within given approximations). Sketch a plot of σ vs. V indicating significance of the special values of V found in this Part.

(f) [**5 points**] Find the values of V where a new bound state forms in the well around zero energy. Compare these values of V with the special values of V found in Part (e). Comment qualitatively on a relation between formation of bound states and scattering at low energies.

Information: The reduced Schrödinger equation for the ℓ-th partial wave $\psi_\ell(r) = u_\ell(r)/r$ is

$$-\frac{\hbar^2}{2m}\frac{d^2 u_\ell(r)}{dr^2} + \frac{\ell(\ell+1)}{2mr^2}\, u_\ell(r) + V(r)u_\ell(r) = E u_\ell(r).$$

Problem II.4

(a) [**5 points**] Consider a three-dimensional isotropic harmonic oscillator of frequency ω_0.

What is the energy of the first excited state, relative to the ground state?
What is the degeneracy of the first excited state?
What is the orbital angular momentum L in the first excited state?

Suppose two spin-1/2 particles of equal masses are subject to the three-dimensional isotropic harmonic oscillator potential. In the rest of the problem, assume that *each particle occupies the first excited energy level* of the harmonic oscillator potential.

(b) [**5 points**] What are the permitted quantum numbers of the total orbital angular momentum $\boldsymbol{L} = \boldsymbol{L}_1 + \boldsymbol{L}_2$, of the total spin $\boldsymbol{S} = \boldsymbol{S}_1 + \boldsymbol{S}_2$, and of the total angular momentum $\boldsymbol{J} = \boldsymbol{J}_1 + \boldsymbol{J}_2$ of the two particles?

(c) [**5 points**] If the two particles are distinguishable, how many linearly independent states do they have?

(d) [**5 points**] Now assume that the two particles are indistinguishable fermions. How many permitted linearly independent states do they have?

(e) [**5 points**] Suppose the two particles are electrons, which experience mutual Coulomb repulsion. Taking into account the Coulomb repulsion qualitatively as a perturbation, what are the quantum numbers S and L giving the lower energy state, and what is the degeneracy of this state?

You do not need to calculate the energy correction due to the Coulomb interaction explicitly. Assume that each particle occupies the first excited energy level of the three-dimensional oscillator and neglect spin-orbit interaction.

Hint: The energy of Coulomb repulsion would be reduced, if the probability for the two particles to be at the same point in space vanishes.

Problem II.5

In this problem, we study stability of heavy nuclei with respect to β-decay. Consider a nucleus with the number of protons Z and the total number of protons and neutrons A. In order to obtain a stability criterion, we first need to derive a formula for the mass M of the nucleus as a function of Z for a fixed A. It is given by a sum of four terms discussed below

$$M(Z, A) = M_1 + M_2 + M_3 + M_4. \tag{1}$$

We treat mass and energy equivalently because of the Einstein relation $E = Mc^2$.

(a) [**2 points**] First, ignore any interaction between protons and neutrons (collectively called nucleons) and write a formula for M_1 in terms of Z, A, and the masses of proton (m_p) and neutron (m_n).

The term M_2 represents binding energy due to nuclear forces, which are *in*dependent of electric charge, so they are the same for protons and neutrons. Thus, we take $M_2 = f(A)$, where f is a complicated unknown function of A. Because this term does not depend on Z, it will be not important for our consideration.

(b) [**4 points**] Protons and neutrons are spin-1/2 fermions. Nuclear forces create a collective attractive potential, where the nucleons occupy energy levels according to the Pauli exclusion principle. How many protons and how many neutrons can be placed in a given energy level? For a nucleus with N completely filled energy levels, what is the relation between Z and A?

(c) [**2 points**] Starting from a nucleus with N completely filled energy levels, suppose we increase $Z \to Z + 1$ while keeping A constant, i.e., transform a neutron into a proton. Following the Pauli exclusion principle, would this process increase or decrease the energy of the nucleus? Repeat the same for the decrease $Z \to Z - 1$ while keeping A constant, i.e., transforming a proton into a neutron. Would this process increase or decrease the energy of the nucleus? Ignore the difference between m_p and m_n here and only focus on occupation of the energy levels dictated by the Pauli principle.

(d) [**2 points**] Treating Z as a continuous variable for large Z, write a simple, lowest-order, smooth (differentiable) function M_3 with an undetermined coefficient a_3 to represent the nuclear energy dependence on Z for small deviations of Z from the optimal number (dependent on A) found in Part (c).

(e) [**4 points**] The term M_4 represents electrostatic energy of the nucleus. Assume that the electric charge of protons is distributed uniformly over a sphere of radius R. Because nuclear matter has an approximately constant density, the volume of the nucleus is proportional to the number of nucleons: $R^3 \propto A$, so $R \propto A^{1/3}$. Using an estimate or dimensional analysis, write down how the term M_4 depends on Z and A. Leave the coefficient, a_4, arbitrary and focus only on the dependence on Z and A.

II.5 (Continued)

Collecting all terms together, you should find the following formula for nuclear mass

$$M(Z,A) = \Big[Zm_p + (A-Z)m_n\Big] + f(A) + a_3\frac{\left(Z-\frac{1}{2}A\right)^2}{A} + a_4\frac{Z^2}{A^{1/3}} \tag{2}$$

with some coefficients a_3 and a_4. The appearance of A in the denominator of M_3 was not derived above, so just take it for granted. Equation (2) is called Weizsäcker's formula.

Now let us discuss stability of a nucleus with respect to β-decay, where an electron or a positron (as well as a neutrino) is emitted from the nucleus. In this process, the number of protons changes $Z \to Z \pm 1$, but the number of nucleons A remains constant.

(f) [**4 points**] Assuming that the electron or positron mass is negligibly small, formulate a stability criterion with respect to β-decay in terms of the energy change $M(Z\pm 1, A) - M(Z, A)$ of the nucleus. Treating Z as a continuos variable, derive conditions on the first and second derivatives, $\partial M/\partial Z$ and $\partial^2 M/\partial Z^2$, for a nucleus to be stable with respect to β-decay.

(g) [**4 points**] Apply the stability criterion derived in Part (f) to Weizsäcker's formula (2) and calculate the stable Z_A for a given A. Do stable nuclei (with A greater than some critical value A_*) have more protons than neutrons or less?

(h) [**3 points**] Derive a formula for the critical parameter A_* introduced in Part (g) and obtain its numerical value using the empirical values $m_n - m_p = 1.3$ MeV and $a_4 = 0.710$ MeV. (The calculated value of A^* turns out to be rather small.)