UNIVERSITY OF MARYLAND
Department of Physics
College Park, Maryland

Ph.D. PHYSICS QUALIFYING EXAMINATION - PART I

January 17, 2013 **9 a.m. - 1 p.m.**

Do any four problems.

Problems I. 1, I. 2, I. 4 and I. 5 are each worth 25 points.

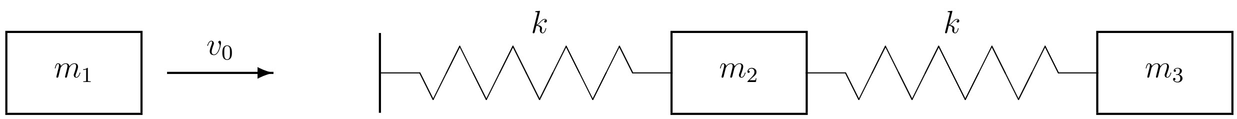
Problem I. 3 Statistical Mechanics is worth 40 points.

Put all answers on your answer sheets.

Be sure your Qualifier ID Number is at the top right corner of each sheet and turn in solutions to four problems only. If five solutions are turned in we will grade # 1 - # 4.

Problem I.1

Consider a system consisting of three particles of masses $m_1 = m$, $m_2 = M$, and $m_3 = m$. At the start, m_1 moves with velocity v_0, collides with the spring extending on the left side of m_2, and sticks to it, thus creating a system of three masses connected by two massless springs of force constant k and equal lengths.



All questions below refer to the system after the collision.

(a) [**3 points**] Write down the Lagrangian of the system.

(b) [**3 points**] Write down the Lagrange equations of motion.

(c) [**8 points**] Determine the eigenfrequencies and eigenvectors (normal modes) of the system. Discuss symmetry of the normal modes.

(d) [**4 points**] What is the center-of-mass velocity of the system?

What is the maximal displacement of the mass m_2 away from the center of mass during the subsequent motion of the system after the collision?

(e) [**7 points**] Calculate the energies allocated to each of the three normal modes after the collision. Verify that their sum is equal to the initial kinetic energy.

Problem I.2

The electric current density $\boldsymbol{J}$ in a superconductor can be expressed in terms of a complex order parameter Ψ and the vector potential $\boldsymbol{A}$:

$$\boldsymbol{J} = -\frac{iq\hbar}{2m}(\Psi^* \boldsymbol{\nabla}\Psi - \Psi \boldsymbol{\nabla}\Psi^*) - \frac{nq^2}{m}\boldsymbol{A}, \qquad \Psi = \sqrt{n}\, e^{i\Phi}. \tag{1}$$

Here m is the mass of a Cooper pair (two electrons), $q = 2e$ is the charge of the Cooper pair, n is the density of Cooper pairs, and the phase Φ is real. For simplicity, assume that n is constant inside the superconductor.

(a) [**6 points**] In order for Eq. (1) for $\boldsymbol{J}$ to be gauge-invariant, how should the phase Φ and the vector potential $\boldsymbol{A}$ change upon a gauge transformation?

(b) [**6 points**] Using Ampère's law and Eq. (1), show that a static, time-independent magnetic field decreases exponentially inside a superconductor over the London penetration depth:

$$\lambda = \sqrt{\frac{m}{nq^2\mu_0}}. \tag{2}$$

(c) [**7 points**] Using Eq. (1) for $\boldsymbol{J}$ and Maxwell's equations with zero charge density, derive a wave equation for the magnetic field $\boldsymbol{B}(\boldsymbol{r}, t)$ inside a superconductor. Show all your work and don't leave any steps out. You should obtain

$$\nabla^2 \boldsymbol{B} - \frac{1}{c^2}\frac{\partial^2 \boldsymbol{B}}{\partial t^2} = \frac{1}{\lambda^2}\boldsymbol{B}. \tag{3}$$

(d) [**6 points**] Consider a plane-wave solution for Eq. (3)

$$\boldsymbol{B}(\boldsymbol{r}, t) = \boldsymbol{B}_0\, e^{i\boldsymbol{k}\cdot\boldsymbol{r} - i\omega t}. \tag{4}$$

(i) Calculate a threshold frequency ω_p for a propagating wave to exist, so that the wave propagates for $\omega > \omega_p$ and exponentially decays as a function of $\boldsymbol{r}$ for $\omega < \omega_p$.

(ii) Find the dispersion relation $\omega(\boldsymbol{k})$ for $\omega > \omega_p$ and show that it is consistent with the notion that photons in a superconductor have a relativistic energy-momentum relation $E(\boldsymbol{p})$ with a mass $m_\gamma = \hbar/\lambda c$.

Maxwell's equations in the absence of charge density in the SI system of units:

$$\boldsymbol{\nabla}\cdot\boldsymbol{B} = 0, \qquad \boldsymbol{\nabla}\cdot\boldsymbol{E} = 0, \qquad \boldsymbol{\nabla}\times\boldsymbol{B} = \mu_0 \boldsymbol{J} + \frac{1}{c^2}\frac{\partial \boldsymbol{E}}{\partial t}, \qquad \boldsymbol{\nabla}\times\boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}.$$

A useful identity for any vector field $\boldsymbol{V}(\boldsymbol{r})$:

$$\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\boldsymbol{V}) = \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{V}) - \nabla^2\boldsymbol{V}$$

Problem I.3

The van der Waals model goes beyond the ideal gas and introduces a repulsive volume exclusion and a short-range attraction to account for interactions between atoms or molecules. A variant of this model (proposed by Dieterici) has the equation of state

$$P(V-b) = RT\exp\left(-\frac{a}{RTV}\right), \tag{1}$$

written here for one mole of a substance. The figure on the next page shows a family of isotherms for this equation. Temperature increases from the bottom to the top curve.

(a) **[8 points]** As shown in the figure, the isothermal compressibility $\kappa_T = -\frac{1}{V}\left(\frac{\partial V}{\partial P}\right)_T$ is always positive at high temperatures, but there is a range of volumes where $\kappa_T < 0$ at low temperatures.
(i) Is the system stable or unstable when $\kappa_T < 0$? Explain why.
(ii) On the figure, connect the points where $\kappa_T = \infty$ by a *dashed* curve, which gives the boundary between the regions with $\kappa_T < 0$ and $\kappa_T > 0$.

(b) **[12 points]** The top point of the dashed curve is the critical point, whose coordinates are (V_c, P_c) on the critical isotherm (having $T = T_c$).
(i) Write the two equations [involving the appropriate derivatives of $P(V)$] that determine the critical point.
(ii) These two equations can be solved for V_c and T_c; however, to save algebra, you may take $V_c = 2b$ and find T_c from one equation and verify that this pair of values (V_c and T_c) also satsfies the second equation. Then find P_c for the Dieterici equation of state (1).

(c) **[4 points]** Find the ratio P_cV_c/RT_c for the Dieterici equation of state and evaluate it on a calculator. For the van der Waals equation of state, this dimensionless combination is 3/8. Which result is closer to the experimental value 0.29 for noble gases?

(d) **[16 points]** Go back to the figure on the next page.

(i) On the isotherm shown by the thick curve on the graph, perform a Maxwell construction and mark the volumes V_1 and V_2 corresponding to the liquid and gas phases that coexist at this temperature. Explain how the Maxwell construction is performed and compare the pressures $P(V_1)$ and $P(V_2)$.

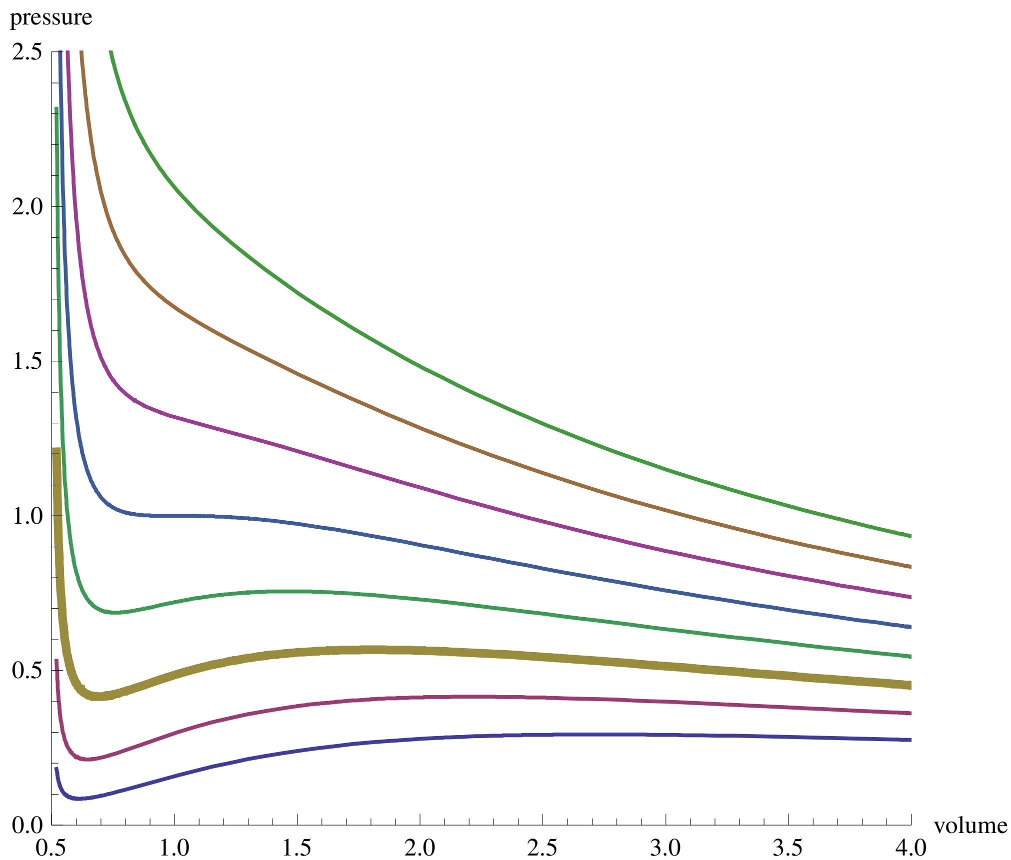
(ii) Suppose the substance is sealed in a container of volume V and has a temperature T such that $V_1(T) < V < V_2(T)$. A) What fraction x_G of the molecules separates into the gas phase? B) What is the pressure in the container: $P(V_1)$, $P(V)$, or $P(V_2)$, where the function P vs. V is given by Eq. (1)?

(iii) Sketch a *solid* curve on the graph indicating the boundary of the region where two-phase liquid-gas coexistence occurs.

(iv) Describe qualitatively what happens in the region between the solid and dashed curves that you have drawn.

Tear out the next page with the graph and submit it with your solution of this problem.
Be sure to write your ***control number*** *on that page.*

I.3 (Continued)



Problem I.4

Although dark matter is believed to constitute about 80% of the matter in the universe, the nature of its constituent particles remains unknown. One interesting possibility is that dark matter is composed of 'dark atoms' that have a ground state and one (or more) excited states. The mass of the excited atom exceeds that of the ground state atom by a small amount Δ. You will investigate whether a collision experiment with visible matter can distinguish between dark atoms and more conventional dark matter that does not possess any excitable internal degrees of freedom.
Consider a dark atom in its ground state of rest mass m that collides with a nucleus of rest mass M at rest. The relativistic energy of the incident dark atom is E and the magnitude of its momentum is p. You may put $c = 1$. Write all answers in terms of these given quantities M, m, E, p, or quantities (such as β) that you have defined in terms of the given quantities. If you find the algebra too complicated you may assume $m = M$ and/or $\Delta \ll m$ in the later parts.

(a) [**5 points**] Find the speed $\beta = v/c$ of the center of momentum of the two particles (dark atom and ordinary nucleus) and obtain an expression for the magnitude of the momenta and for the energies of the incident dark atom and nucleus in the center of momentum frame.

(b) [**7 points**] Consider first elastic scattering, so that the dark atom remains in its ground state. Obtain an expression for the kinetic energy of the recoiling nucleus in the lab frame as a function of some scattering angle θ you define.

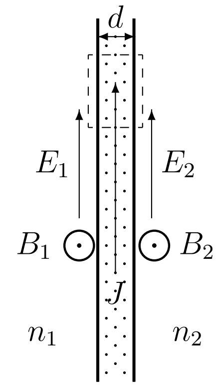
(c) [**3 points**] What are the largest and smallest possible values of the relativistic energy of the recoiling nucleus in the lab frame, as θ is varied?

(d) [**7 points**] Next consider inelastic scattering, when the rest mass of the dark atom after collision is $m + \Delta$. What is the smallest ('threshold') value E_{thr} for E such that inelastic scattering into this excited state is kinematically allowed?
Hint: what is the relative velocity between the masses $m+\Delta$ and M after the collision?

(e) [**2 points**] Find the kinetic energy $K = E_{\text{rec. nucl.}} - M$ of the recoil nucleus after the theshold scattering.

(f) [**1 point**] Assume a broad spectrum of incident dark matter energies that extends well beyond E_{thr}. Also assume that the inelastic scattering of dark atoms peaks near the threshold values you found in (d) and (e). Conventional dark matter, on the other hand, always scatters elastically, since it possesses no excited states. In view of what you found in (c) about the minimum elastically scattered recoil energy, how might you infer from the nuclear recoil spectrum that dark matter is composed of dark atoms rather than of conventional dark matter?

Problem I.5



Two optically transparent materials are separated by a planar thin film of thickness d, which is shown in the Figure as the dotted region extending perpendicular to the page. The film is a metal of high electric conductivity σ. The materials on the left and right sides have the indices of refraction n_1 and n_2, respectively.

Consider a linearly-polarized electromagnetic plane wave of frequency ω that is normally incident on the interface from the left. Assume that the film thickness d is much less than the wavelength of light and the penetration depth of the metal. In Gaussian units, take the conductivity to be large compared to the frequency, $\sigma \gg \omega$.

(a) [**10 points**] In the conducting film, the current density $\boldsymbol{J}$ is given by Ohm's law, $\boldsymbol{J} = \sigma \boldsymbol{E}$, where $\boldsymbol{E}$ is the electric field. In the optically transparent materials, the magnetic field $\boldsymbol{H}$ is approximately equal to the magnetic induction $\boldsymbol{B}$.

Using these relations and integrating Faraday's and Ampère's laws along the closed path shown by the dashed line in the Figure (or one rotated by 90°), prove that the boundary conditions relating the tangential components of the electric and magnetic fields at the interface are

$$E_1 = E_2, \qquad B_1 = B_2 + \kappa E_2, \qquad \kappa = \frac{4\pi\sigma d}{c}.$$

Direction: Take the limit $d \to 0$, while keeping σd and κ finite.

(b) [**10 points**] Using the above boundary conditions and the relations $B = nE$ in the transparent materials, calculate the intensity reflection coefficient from the interface in terms of n_1, n_2, and κ.

(c) [**5 points**] What are the conditions for the metallic film to act as an anti-reflecting coating?

Faraday's and Ampère's laws in the Gaussian units are

$$\boldsymbol{\nabla} \times \boldsymbol{E} = -\frac{1}{c}\frac{\partial \boldsymbol{B}}{\partial t}, \qquad \boldsymbol{\nabla} \times \boldsymbol{H} = \frac{4\pi}{c}\boldsymbol{J} + \frac{1}{c}\frac{\partial \boldsymbol{D}}{\partial t}.$$