UNIVERSITY OF MARYLAND
Department of Physics
College Park, Maryland

Ph.D. PHYSICS QUALIFYING EXAMINATION - PART I

August 29th, 2013 **9 a.m. - 1 p.m.**

Do any four problems. Each problem is worth 25 points.

Put all answers on your answer sheets.

Be sure your Qualifier ID Number is at the top right corner of each sheet and turn in solutions to four problems only. If five solutions are turned in we will grade # 1 - # 4.

Problem I.1

Consider a particle of mass m confined to a two-dimensional plane with polar coordinates (r, ϕ), in the presence of an upside-down harmonic oscillator potential $V(r) = -\frac{1}{2}\kappa r^2$.

(a) [**5 points**] Identify two conserved quantities for the particle's motion, and write expressions for them.

(b) [**5 points**] Now assume the particle has an electric charge q, and there is a uniform and time-independent magnetic field of strength B, normal to the plane of motion. Write the Lagrangian L and the energy E for the particle as functions of $(r, \dot{r}, \phi, \dot{\phi})$. [A convenient vector potential for the magnetic field in this problem is $\boldsymbol{A} = (Br/2)\hat{\phi}$.]

(c) [**5 points**] Identify the conserved quantities for the motion, and write expressions for them.

Find the effective potential $V_{\text{eff}}(r)$ for the radial motion in the presence of the magnetic field, for any fixed value of the conserved variable p_ϕ conjugate to the angle ϕ.

Sketch $V_{\text{eff}}(r)$ in the limit of small and of large B.

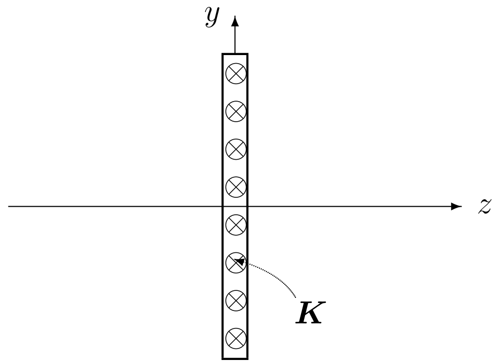
(d) [**5 points**] Show that the motion is stable (i.e. the particle remains for all time within a fixed distance from the origin) provided $|B|$ is larger than some minimum value B_0, and find that value.

(e) [**5 points**] Find the particle position (r, ϕ) as a function of time t for all trajectories that pass through the origin at time $t = 0$, assuming

(i) $|B| < B_0$,
(ii) $|B| = B_0$, and
(iii) $|B| > B_0$.

Problem I.2

An infinite conducting, current-carrying plate lies in the x-y plane (see Figure). Suppose there is a steady current in the plate in the $+x$ direction. Let the current per unit y-length be K.



(a) [**3 points**] Using Ampère's Law, $\oint \boldsymbol{B}\cdot d\boldsymbol{l} = \mu_0 I_{enc}$, find the steady magnetic field B_y in terms of K on either side of the plate. Assume that the fields associated with the current are appropriately reflection symmetric about the plate. Make a sketch of $\boldsymbol{B}$ for all z, using the Figure as reference.

Now suppose instead that there is time dependence: the current and fields vanish for $t < 0$, and K is turned on suddenly at $t = 0$. That is, $K = K_0\Theta(t)$ where Θ is the step function, $\Theta(t) = \left\{\begin{smallmatrix} 0 & t<0 \\ 1 & t>0 \end{smallmatrix}\right.$. Consider now a snapshot of the system at some time $t > 0$:

(b) [**4 points**] *Using only qualitative reasoning*, in particular using causality, make a sketch of the snapshot of $\boldsymbol{B}$ in the Figure at time $t > 0$. Mark clearly any characteristic lengths in your sketch. (Note that the plate is of infinite extent; thus, the problem is still one-dimensional in z, but the $\boldsymbol{B}$ field may have structure in that direction.)

(c) [**4 points**] Do you think the total energy stored in the electromagnetic fields increases or decreases with time? Based on this, consider *qualitatively* the Poynting flux and deduce if there should be any associated electric fields $\boldsymbol{E}$. If so, include $\boldsymbol{E}$ in your sketch, with the proper signs.

From Maxwell equations, the vector potential $\boldsymbol{A}(z,t)$ can be shown to satisfy the wave equation

$$\left(\partial_z^2 - \frac{1}{c^2}\partial_t^2\right)\boldsymbol{A} = -\mu_0 \boldsymbol{j},$$

where the current density is

$$\boldsymbol{j} = K\delta(z)\Theta(t)\ \hat{x}.$$

for our problem above. The solution for $\boldsymbol{A}$ can be constructed from a Green function, $G(z,z',t,t')$, according to

$$\boldsymbol{A}(z,t) = -\mu_0 \int_{-\infty}^{\infty} dt' \int_{-\infty}^{\infty} dz'\, G(z,z',t,t')\ \boldsymbol{j}(z',t').$$

The Green function, in turn, for outgoing wave boundary conditions (i.e., causality), can be found to be

$$G = \frac{c}{2}\Theta(t_r - t') \qquad \text{where} \quad t_r = t - \frac{|z-z'|}{c}.$$

(d) [**5 points**] Find the non-zero component of the vector potential $\boldsymbol{A}(z,t)$ for all (z,t). Leave your answer in terms of an integral over t'.

I.2 (Continued)

(e) **[3 points]** The electromagnetic fields can be calculated from $\boldsymbol{A}(z,t)$ according to the relations given below. (There is no scalar potential φ in this problem, so ignore it.) Calculate $B_y(z,t)$ [perform any differentiations by commuting across the integral]. Describe the waveform in z for given t clearly, and compare with your snapshot above.

Hint: Consider separately the cases $t_r < 0$ and $t_r > 0$.

(f) **[3 points]** Likewise, calculate $\boldsymbol{E}(z,t)$. Compare with your snapshot above.

(g) **[3 points]** For $z > 0$, calculate the energy per unit plate area stored in the electromagnetic fields at time t. Calculate the Poynting energy flux per unit plate area. Confirm that the power input into the electromagnetic energy equals the Poynting flux.

(Expressions for the electromagnetic energy density, u, and Poynting flux, $\boldsymbol{S}$, are given below.)

Useful equations

$$\Theta[f] = \begin{cases} 0 & f < 0 \\ 1 & f > 0 \end{cases} \qquad \frac{d}{dt}\Theta[f(t)] = \delta(f)\frac{df}{dt}$$

$$\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}, \qquad \boldsymbol{E} = -\boldsymbol{\nabla}\varphi - \frac{\partial \boldsymbol{A}}{\partial t}$$

$$u = \frac{1}{2}\left(\epsilon_0 E^2 + \frac{B^2}{\mu_0}\right), \qquad \boldsymbol{S} = \frac{\boldsymbol{E} \times \boldsymbol{B}}{\mu_0}$$

Problem I.3

Equilibrium is a central concept in Statistical Physics.

(a) [**6 points**] Give a definition of equilibrium for

  (a) an isolated system

  (b) a pair of interacting systems

(b) [**6 points**]

  (a) For a closed isolated system, give two examples of experimentally measurable thermodynamic variables, their definitions, and what equilibrium implies about them.

  (b) For a system that can exchange particles, give one more example of an experimentally measurable thermodynamic variable and its definition.

(c) [**7 points**] A colleague gives you a crystalline solid sample and asks you to examine its electronic properties (e.g. voltage V versus current I).

  (a) What are the thermodynamic variables and potentials that are relevant to your measurement?

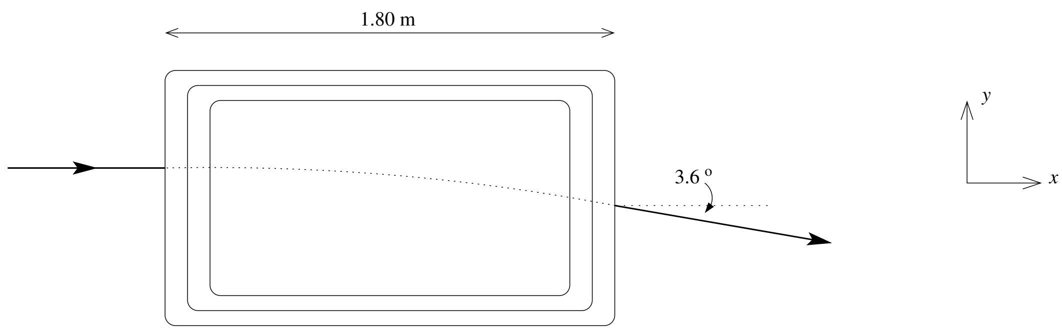
  (b) As an experimentalist, why should you concern yourself with the sample equilibrium?

(d) [**6 points**] A laboratory is never in equilibrium, in the sense of an isolated system. Give four examples of likely nonequilibrium conditions for a scientific laboratory in this building (or another campus building if you prefer).

Problem I.4

A *magnetic spectrometer* can be used to determine properties of charged particles. In this problem we assume that the particle trajectories outside of the magnet are measured accurately by certain detectors (such as silicon pixel or strip detectors, or drift chambers).

The Figure below (not to scale) shows a particle traveling through such a magnet. In the right-handed coordinate system indicated, the particle is traveling in the x-y plane, with no z component to its velocity. The magnet has a magnetic field that is approximated to be strictly constant in strength and direction inside the magnet, and strictly zero outside the magnet (even though real magnets have "fringe fields" near their edges).



(a) **[3 points]** Suppose you observe that a particle enters the magnet parallel to the x axis and is deflected in the $-y$ direction by the time it exits, as shown in the figure. Is the particle's total energy (in the reference frame of the magnet) different after exiting the magnet than it was before entering the magnet? Explain.

(b) **[4 points]** Show that the trajectory of the particle in the uniform field of the magnet is a circular arc. Express the magnitude p of the particle's momentum in terms of its electric charge q, the magnetic field $\boldsymbol{B} = B_0\hat{\boldsymbol{z}}$, and radius r of the trajectory, showing that we can infer p from the observed r if we know q and B_0.

Justify that your result holds for the momentum of *relativistically* moving particles

(c) **[5 points]** Suppose that the magnet shown in the figure above is 1.80 m long and has a magnetic field strength $B_0 = 0.370\,\mathrm{T}$, and we know that the particle has charge $e = 1.602 \times 10^{-19}\,\mathrm{C}$, and is deflected by an angle of 3.60°. What is the momentum of the particle?

Hint: the tesla (T) is the standard SI unit for magnetic field. Express your answer in MeV/c units; note that $1\,\mathrm{eV} = 1.602 \times 10^{-19}\,\mathrm{J}$, so $1\,\mathrm{MeV}/c = 5.344 \times 10^{-22}\,\mathrm{kg \cdot m/s}$.

(d) **[1 point]** Another useful tool is a Cherenkov detector. A charged particle moving at speed v through a medium with index of refraction n produces Cherenkov light in a cone with a half-angle given by $\cos\theta = c/nv$.

Suppose the particle described above passes through a Cherenkov detector filled with high-pressure gas giving $n = 1.047$, and the cone half-angle is measured to be $\theta = 11.6°$. From this observation, what is the particle's speed, expressed as a fraction of the speed of light?

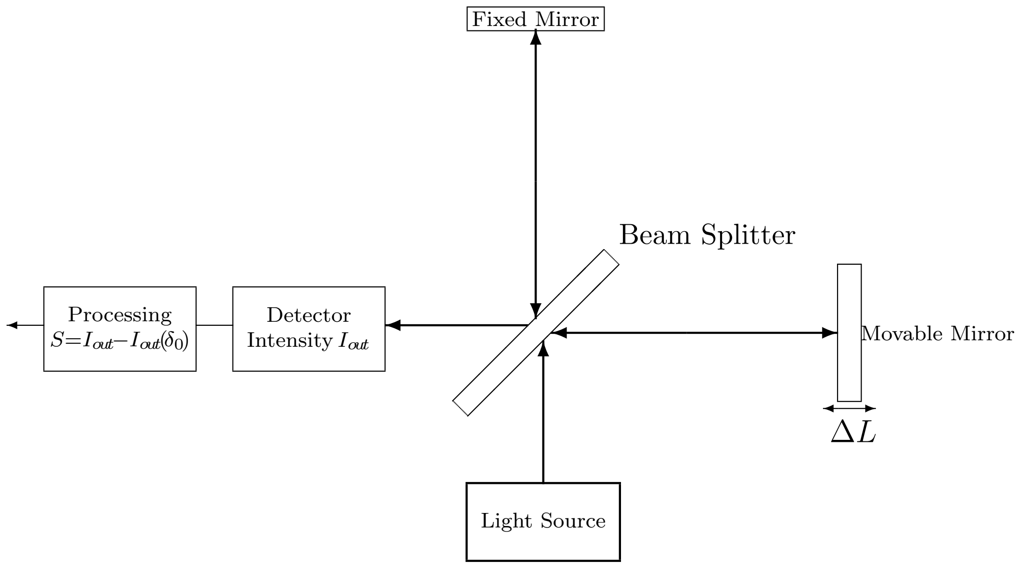
I.4 (Continued)

(e) [**4 points**] Combining your answers from the previous parts, find the rest mass of the particle in MeV/c^2 units,

(f) [**4 points**] From the particle's point of view, *i.e.* in its reference frame and using its clock, how much time (in seconds) does it spend inside the magnet?

(g) [**4 points**] In the "coincident" inertial frame, which has the same velocity as that of the particle's initial, straight-line motion, the particle is initially motionless, of course. As the particle enters the spectrometer's field it still has $\boldsymbol{v} = 0$ but is immediately deflected sideways. In this coincident inertial frame, what force causes the deflection?

Problem I.5



The Figure shows a Michelson interferometer in which two coherent light waves, derived from the same monochromatic source, are combined into one wave of intensity I_{out}. This intensity is measured and processed into a signal S that is sensitive to small displacements ΔL of a movable mirror relative to some reference position. You will determine the optimum reference configuration according to two different criteria, and explain any difference.

(a) [**5 points**] The electric fields at the detector from the two light waves are polarized in the same way and have amplitudes E_f and E_m from reflection by the fixed and movable mirrors, respectively. There is a phase difference δ between the two waves.

Show that the dependence of the output intensity $I_{out}(\delta)$ of the Michelson interferometer on the phase difference δ has the form

$$I_{out}(\delta) = A + B\cos^2\frac{\delta}{2} \tag{1}$$

where A and B are positive constants. Find A and B in terms of E_f and E_m.

(b) [**2 points**] Express the phase difference δ caused by motion of the movable mirror in terms of its displacement ΔL and the wavelength λ of the light source.

(c) [**5 points**] The fringe visibility V in the output of the interferometer gives a quantitative indication of the contrast of interference,

$$V = \frac{I_{max} - I_{min}}{I_{max} + I_{min}} \tag{2}$$

In terms of B, what values of A in Eq. (1) are needed to obtain visibilities $V = 0.5$ and 1.0?

Plot I_{out} as a function of phase difference δ for these two values of V. Label both of the axes carefully.

I.5 (Continued)

The output intensity I_{out} of the interferometer can be used to measure very small displacements in the movable mirror. This is the principle that LIGO uses to detect gravitational waves. Since the wave's local metric oscillates in time, so does ΔL and δ. Assume that the phase δ oscillates with small amplitude ϵ about some reference phase δ_o, $\delta = \delta_o + \epsilon \sin \omega t$. (Here $\omega \ll c/2\pi\lambda$.) This leads to an oscillation of I_{out} about $I_{out}(\delta_o)$. The amplitude of this oscillation is the measured signal S, so to $O(\epsilon)$,

$$S = \epsilon \left. \frac{dI_{out}(\delta)}{d\delta} \right|_{\delta=\delta_o} . \tag{3}$$

The reference phase δ_0 can be adjusted to an optimum value, for example by changing the position of the fixed mirror.

(d) **[3 points]** For what value of δ_o is $|S|$ maximized? What is S at this point?

(e) **[6 points]** Now consider that in a real measurement you have various noise contributions, so you may instead want to maximize the Signal to Noise ratio S/N. Often the largest noise is the shot noise, $N \sim \sqrt{I_{out}(\delta_o)}$. Suppose the visibility is close to unity so that in Eq. (1), $A \ll B$. For this case, what value of δ_o maximizes $|S/N|$ (in terms of A)? What is $|S|$ and (up to a factor) $|S/N|$ at this point? Interpret this result.

(f) **[4 points]** For small A, parts (d) and (e) give different optimum operating points. Discuss this result.

Useful trigonometric relations:

$$2\cos^2 A = 1 + \cos 2A$$

$$\cos(A+B) = \cos A \cos B - \sin A \sin B$$

$$\sin A + \sin B = 2\cos\left[\frac{A-B}{2}\right] \sin\left[\frac{A+B}{2}\right]$$

$$\sin(A+B) = \sin A \cos B + \cos A \sin B$$