**UNIVERSITY OF MARYLAND**
**Department of Physics**
**College Park, Maryland**

# Ph.D. PHYSICS QUALIFYING EXAMINATION - PART I

**January 23, 2014** **9 a.m. - 1 p.m.**

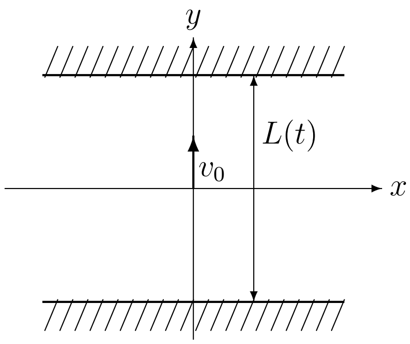
**Do any four problems. Each problem is worth 25 points.**

**Put all answers on your answer sheets.**

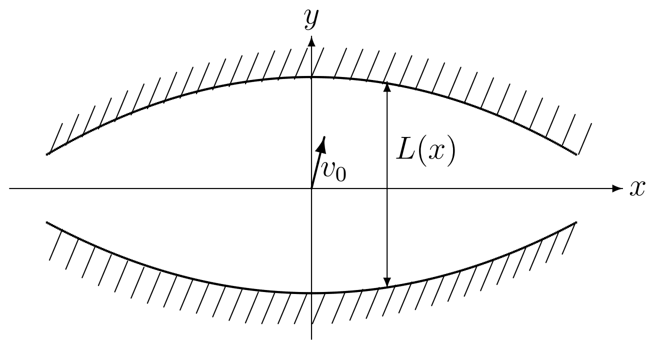
**Be sure your Qualifier ID Number is at the top right corner of each sheet and turn in solutions to four problems only. If five solutions are turned in we will grade # 1 - # 4.**

## Problem I.1

The figures below show two examples of conditions that change slowly enough that the resulting changes in motion of an elastically bouncing particle may be treated adiabatically. In figure (a) the particle moves only in the $y$-direction and reflects from moving walls. In figure (b), $v_x \ll |v_y|$. There is no gravity.




**(a)**




**(b)**

(a) **[4 points]** The two smoothly converging walls in Figure (a) move at the same speed and have a time-dependent separation $L(t)$. How does the particle speed $v$ change, due to the reflection from the moving walls, in one complete (back-and-forth) bounce? Note: The reflection on the moving wall is elastic *in the frame of that wall.*

(b) **[5 points]** When the speed of the ball is much greater than the moving wall's speed $v_w = -\frac{1}{2}dL/dt$, then on a time scale $L/v_w$ you can regard the change in $v$ as a continuous process. Under this assumption, write a differential equation for $dv/dt$.

(c) **[5 points]** Solve the differential equation and obtain an adiabatic invariant of this motion.

(d) **[4 points]** Now consider the motion of a ball bouncing elastically between two stationary curved walls as shown in Figure (b). The separation of the walls is given by $L(x) = L_0(1 - x^2/a^2)$. Assume that the ball crosses the midpoint with velocity $v_0$ (where $v_0^2 = v_{0x}^2 + v_{0y}^2$), and define the angle $\theta_0 = \tan^{-1}(v_{0x}/v_{0y})$. Assuming that $v_{0x}$ is sufficiently small (how small to be addressed later), there is an adiabatic invariant [as in part (c)], and a conserved quantity associated with the motion. What are these time-independent quantities? Describe qualitatively the motion of the ball.

(e) **[5 points]** Using the quantities found in (d), derive an equation for the motion of the ball in the $x$ direction, again assuming $v_{0x}$ is sufficiently small and $x^2/a^2 \ll 1$. Characterize the motion in $x$ in terms of its repetition frequency $\omega$ and its maximum excursion $x_0$. Write your answers in terms of $v_0$, $a$, $L_0$, and $\theta_0$.

(f) **[2 points]** Derive the inequality under which the two quantities described in (d) are (nearly) constant. Express your answer in terms of $a$, $L_0$, and $\theta_0$.

## Problem I.2

A pulsar emits regularly spaced pulses of electromagnetic waves. The pulses pass through interstellar space containing a plasma with free electron density $n$ to reach a radio telescope on Earth.

(a) **[4 points]** The free electrons in the plasma oscillate as the wave propagates, inducing a charge current density $\mathbf{j}$. Using the Maxwell equations, derive the partial-differential wave equation for EM propagation through the plasma.

(b) **[4 points]** Using Newton's second law for an electron responding to the Coulomb force (ignore the smaller magnetic field), derive an expression for $\frac{\partial \mathbf{j}}{\partial t}$ due to the EM wave traveling through the plasma.

(c) **[4 points]** Assuming the EM wave is linearly polarized of the form $E_0 e^{i(kz-\omega t)}\hat{x}$, identify the dispersion relation $\omega(k)$ for this wave.

(d) **[4 points]** By analogy with the relation in vacuum $\epsilon = \frac{c^2k^2}{\omega^2}$, show that the effective relative permittivity of the plasma is

$$\epsilon = 1 - \frac{\omega_p^2}{\omega^2}. \tag{1}$$

Derive the expression for the "plasma frequency" $\omega_p$ in terms of the charge $e$, mass $m$, and free-electron plasma density $n$.

(e) **[3 points]** Find the phase and group velocity of the radio wave pulse through the plasma.

(f) **[3 points]** The arrival times of these pulses on Earth are delayed for lower frequencies. Suppose that the pulsar emits two pulses with different frequencies $\omega_1, \omega_2 > \omega_p$ simultaneously. Find an expression for the delay time $\Delta\tau$ between the two pulses in terms of the frequencies, the distance $L$ between the pulsar and Earth, and the free electron plasma density $n$.

(g) **[3 points]** Assume $\omega_1, \omega_2 \gg \omega_p$. For $\omega_1 = 3 \times 10^9$ rad/s, $\omega_2 = 4 \times 10^9$ rad/s, $L =$1000 lightyears, and $\Delta\tau =$100 ms, find the interstellar free electron plasma density in $\text{cm}^{-3}$. Use $c = 3 \times 10^{10}$ cm/s, $m = 5 \times 10^5$ eV/$c^2$, and $\epsilon_0 = 10^{-11}$ C/(V$\cdot$ m). Note that $e \approx 10^{-19}$ C.

**Maxwell equations:**

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \times \mathbf{B} = \mu_0 \left[\mathbf{j} + \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}\right]$$

## Problem I.3

This problem deals with the quantum to classical crossover in a three-dimensional Fermi gas.

Consider a non-interacting gas of $N$ spin-1/2 fermions each of mass $m$ confined in a volume $V$. The number density is $n = N/V$.

(a) **[10 points]**

- (i) If the gas were a classical gas at temperature $T$, what is its energy?
- (ii) In terms of the Fermi energy (or Fermi temperature) what is the ground state ($T = 0$) energy of the Fermi gas? In terms of number density and mass what is this energy?
- (iii) At what temperature does the classical energy equal the zero temperature quantum energy?
- (iv) Using these results, sketch the energy as a function of $T$.

(b) **[8 points]**

- (i) In terms of the thermal deBroglie wavelength, what is the condition for the classical to quantum crossover? Physically explain this result.
- (ii) Assume that the gas is in an initial volume, $V_i$, and its temperature is so low that the gas behaves as a degenerate gas. Let the gas expand irreversibly at constant temperature to a final volume, $V_f$. Calculate a minimum value of $V_f$ for the gas to behave classically.

Now assume that there are impurities in the Fermi gas and that the mean-free time between gas particle-impurity collision is $\tau$.

(c) **[7 points]**

- (i) A particle current is caused by a density gradient with the proportionality constant being the diffusion coefficient. From this *law* determine the units of the diffusion coefficient, $D$.
- (ii) In the degenerate limit, why is the Fermi velocity, $v_F$, the relevant velocity for transport? Estimate the quantum diffusion coefficient using this and dimensional analysis,
- (iii) What would the corresponding classical expression be?
- (iv) Using these results, sketch $D$ as a function of temperature.

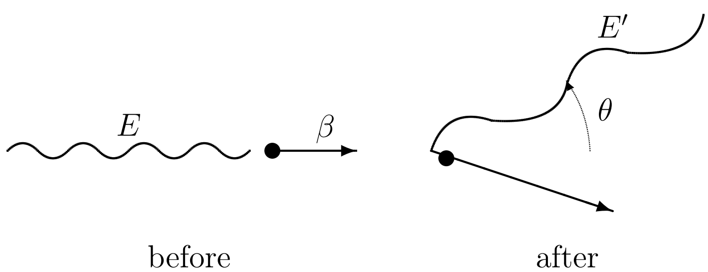
**Problem I.4**

Consider scattering of a photon of frequency $\nu$ and wavelength $\lambda$ off an electron of mass $m$, so weakly bound to an atom that it can be assumed free and at rest.

(a) **[2 points]** Ignoring spin, characterize the energy-momentum 4-vector of each particle in terms of invariants.

(b) **[11 points]** Find the wavelength $\lambda'$ of the photon after the scattering in terms of $\lambda$, $m$, and the scattering angle $\theta$ of the photon in the rest frame of the initial electron.

If the electron is initially not at rest, it can transfer some energy to the photon under certain conditions, which you will investigate. For simplicity assume that the initial photon and the electron move along the same line. The initial photon energy is $E$, the electron energy and momentum are $\gamma mc^2$ and $\beta\gamma mc$, and the scattering angle is $\theta$.




(c) **[3 points]** Show that the photon cannot gain energy ($E' < E$) if the initial electron moves in the same direction as the initial photon.

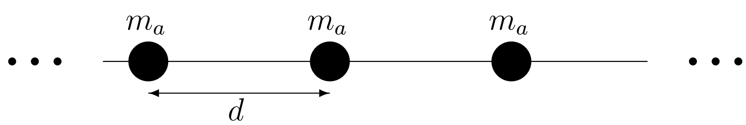
(d) **[3 points]** Show that the photon energy is unchanged for forward scattering ($\theta = 0$).

(e) **[3 points]** Therefore the most promising configuration for energy gain is head-on collision ($\beta < 0$) and backscattering ($\theta = -\pi$). Work out $\Delta E/E = (E' - E)/E$ in terms of $E$, $\beta$ and $\gamma$, and show that $\Delta E$ becomes positive when the electron momentum exceeds $E/c$.

(f) **[3 points]** When the electron energy is large ($\gamma \gg 1$) and the initial photon energy is small ($E \ll mc^2/\gamma$), show that $\Delta E/E \sim 4\gamma^2$ – a very large increase.

## Problem I.5

An infinite one-dimensional array of identical particles of mass $m_a$ lie along the $x$-axis, and are all equally spaced a distance $d$ apart when in equilibrium. Consider only the two nearest-neighbor interactions and model the force on the particle at position $n$ due to interaction with the particle at $n+1$ via Hooke's law as $F_{n \to n+1} = \beta(u_{n+1} - u_n)$, where $u_n$, the only degree of freedom, is the displacement of the $n^{\text{th}}$ particle along $x$ from its equilibrium position $x_n$.

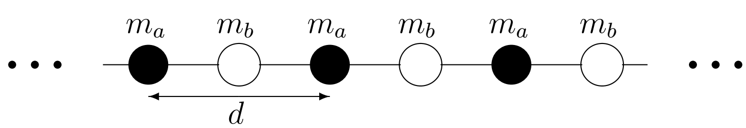



(a) **[2 points]** Using Newton's second law, write down the explicit equation of motion for the displacement of the $n^{\text{th}}$ particle, $u_n$.

(b) **[5 points]** To find the normal modes of this system, substitute a wavelike response $u_n = u(k)e^{i(kx_n - \omega t)}$, and find the dispersion relation $\omega(k)$ of this wave.

(c) **[3 points]** Show that the low-frequency excitations have an approximately linear dispersion relation. What is the group velocity?

Now consider a similar infinite one-dimensional array, but here there are *two* types of particles: those with mass $m_a$ alternate with those of mass $m_b$ along the array as shown in the figure below, and the spring constant for interactions between neighboring particles is now $2\beta$.




(d) **[2 points]** Write down the explicit equations of motion for the displacement of two neighboring particles, $u_n^{a,b}$ and $u_{n+1}^{b,a}$.

(e) **[5 points]** To find the normal modes of this system, substitute a wavelike response $u_n^{a,b} = u^{a,b}(k)e^{i(kx_n - \omega t)}$. Convert these two algebraic equations into a 2×2 matrix eigenvalue problem.

(f) **[4 points]** Find the (two-valued) dispersion relation $\omega(k)$ of this wave by solving for the eigenvalues of the matrix above. Sketch both branches of $\omega(k)$, and show that when $m_b = 0$, the lower branch reduces to the dispersion found in part (b).

(g) **[4 points]** By examining the relative phase between adjacent particles for $k \to 0$ (long-wavelength) excitations, identify the qualitative difference between the modes of the two branches.