UNIVERSITY OF MARYLAND

Department of Physics

College Park, Maryland

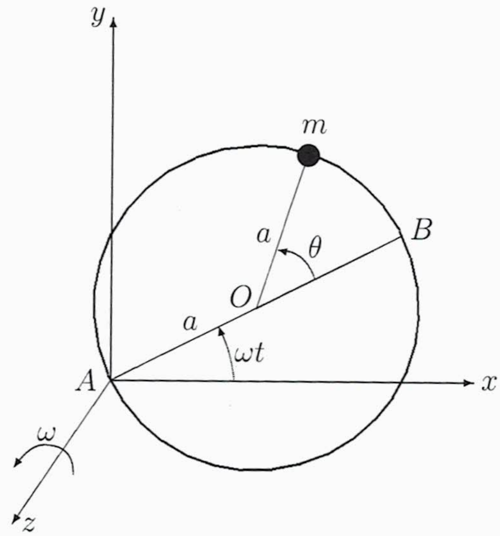
PHYSICS Ph.D. QUALIFYING EXAMINATION

PART A

August 14, 2024	**10:00 am – 12:00 pm**
August 15, 2024	**10:00 am – 12:00 pm**

Problem A.1

A bead of mass m slides on a hoop of radius a that lies in the xy plane and rotates with constant angular velocity ω about the perpendicular axis z passing through point A at the edge of the hoop, as shown in the Figure. The position of the bead is determined by the angle θ between the diameter line AB and the line pointing to the bead from the center O of the hoop. The problem ignores gravity and friction.



(a) **[5 points]** Using the time-dependent angle $\theta(t)$ as a generalized coordinate, obtain the Cartesian velocities, $\dot{x}(t)$ and $\dot{y}(t)$, of the bead.

Suggestion: It may be helpful to obtain the Cartesian coordinates, $x(t)$ and $y(t)$, of the bead first.

(b) **[5 points]** Find the kinetic energy, $T(\theta, \dot{\theta})$, and the Langrangian, $\mathcal{L}(\theta, \dot{\theta})$.

Hint: $\cos\alpha\cos(\alpha+\beta) + \sin\alpha\sin(\alpha+\beta) = \cos\beta$.

(c) **[5 points]** Derive the Lagrange equation of motion for the angle $\theta(t)$. Comment on the simple physical system that this equation of motion resembles.

(d) **[3 points]** Find the stationary points $\theta(t) = \theta_s = \text{const}$ of the equation of motion, and indicate them on the Figure. Are they stable or unstable?

For the stable point(s), find the characteristic frequency of oscillation.

(e) **[4 points]** Find the integral of motion $E(\theta, \dot{\theta})$, such that $dE(\theta, \dot{\theta})/dt = 0$.

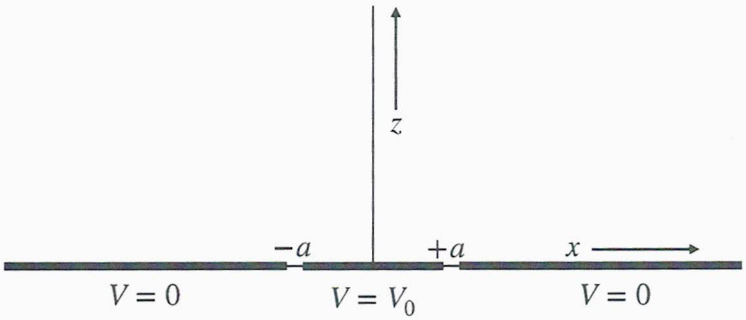
Hint: Multiply the equation of motion by $\dot{\theta}$ and either integrate over time or show that the resulting equation has the form $dE(\theta, \dot{\theta})/dt = 0$.

(f) **[3 points]** Suppose the bead is initially located at point A and acquires an infinitesimally small initial angular velocity.

Find the angular velocity $\dot{\theta}$ of the bead when it arrives to point B.

Problem A.2

As shown in the figure, an infinite conducting plate lies in the x-y plane at $z = 0$, where the y-axis is perpendicular to the page. The plate is sliced at $x = \pm a$, and the segment $|x| < a$ is maintained at a constant potential V_0, while the rest of the conductor is grounded at $V = 0$.



(a) **[2 points]** We are interested in the electric potential $V(x, y, z)$. Given the symmetry of the configuration, it obviously does not depend on y, so can be written as $V(x, z)$. What is the differential equation satisfied by $V(x, z)$ for $z > 0$?

(b) **[1 point]** What is $V(x, z)$ at large z?

(c) **[5 points]** Using separation of variables in Cartesian coordinates, write the basis functions satisfying the differential equation for $V(x, z)$ in Part (a) and the boundary condition in Part (b) in the region $z > 0$.

(d) **[2 points]** Next, consider the given boundary condition at $z = 0$. Sketch and write a formula for $V(x, 0)$ at $z = 0$ as a function of x.

(e) **[5 points]** Write $V(x, z)$ as an integral over the basis functions from Part (c) with some coefficients and determine these coefficients from the boundary condition in Part (d). (There is *no* need to evaluate the resulting integral.)

(f) **[5 points]** Using $V(x, z)$ from Part (e), calculate the z-component of the electric field $E_z(x, z)$ by *evaluating the involved integral.*

(g) **[5 points]** Taking the limit $z \to 0^+$ in Part (f), obtain the induced surface charge density $\sigma(x)$ on the plate and make a sketch of $\sigma(x)$.

Hint: Use the symmetry between the regions $z > 0$ and $z < 0$.

Problem A.3

Consider a two-dimensional Hilbert space corresponding to spin 1/2. In this basis, any Hermitian operator can be represented as a linear combination of the unit matrix $\hat{1}$ and the three Pauli matrices $\hat{\sigma}_j$ given at the bottom of the page, which satisfy the algebra

$$\hat{\sigma}_j \hat{\sigma}_k = \delta_{jk} \hat{1} + i\epsilon_{jkl}\, \hat{\sigma}_l. \tag{1}$$

Here the indices j, k, and l take the values x, y, and z; δ_{jk} is the Kronecker symbol, and ϵ_{jkl} is the antisymmetric Levi-Civita symbol. Summation over the repeated index l is implied.

(a) **[4 points]** Let us introduce a unit vector $\boldsymbol{n} = (n_x, n_y, n_z)$ satisfying $\boldsymbol{n}^2 = 1$ and construct the operator $\boldsymbol{n} \cdot \hat{\boldsymbol{\sigma}}$, where $\hat{\boldsymbol{\sigma}} = (\hat{\sigma}_x, \hat{\sigma}_y, \hat{\sigma}_z)$ is the vector of Pauli matrices.

Using Eq. (1), calculate the operator $(\boldsymbol{n} \cdot \hat{\boldsymbol{\sigma}})^2$ and find its eigenvalues.

(b) **[4 points]** Let us introduce the two eigenvectors $|\boldsymbol{n}, +\rangle$ and $|\boldsymbol{n}, -\rangle$ of the operator $\boldsymbol{n} \cdot \hat{\boldsymbol{\sigma}}$, such that $\boldsymbol{n} \cdot \hat{\boldsymbol{\sigma}} |\boldsymbol{n}, +\rangle = \lambda_+ |\boldsymbol{n}, +\rangle$ and $\boldsymbol{n} \cdot \hat{\boldsymbol{\sigma}} |\boldsymbol{n}, -\rangle = \lambda_- |\boldsymbol{n}, -\rangle$.

Using your result from Part (a), find the eigenvalues λ_+ and λ_-, with + denoting the larger of the two.

(c) **[4 points]** Let us introduce the projection operators $\hat{P}_+^{\boldsymbol{n}} = |\boldsymbol{n}, +\rangle\langle \boldsymbol{n}, +|$ and $\hat{P}_-^{\boldsymbol{n}} = |\boldsymbol{n}, -\rangle\langle \boldsymbol{n}, -|$ for the two eigenstates from Part (b).

Show that these projection operators satisfy the relation $\hat{P}^2 = \hat{P}$. Find the eigenvalues and the eigenstates of $\hat{P}_+^{\boldsymbol{n}}$ and $\hat{P}_-^{\boldsymbol{n}}$.

(d) **[4 points]** Given the results of Parts (b) and (c), write the projection operators $\hat{P}_+^{\boldsymbol{n}}$ and $\hat{P}_-^{\boldsymbol{n}}$ as suitable linear combinations of the operators $\hat{1}$ and $\boldsymbol{n} \cdot \hat{\boldsymbol{\sigma}}$.

(e) **[5 points]** Consider a product $\hat{P}_+^{\boldsymbol{m}} \hat{P}_+^{\boldsymbol{n}}$ of the two projection operators for the unit vectors $\boldsymbol{n}$ and $\boldsymbol{m}$. Show that it can be written in the form

$$\hat{P}_+^{\boldsymbol{m}} \hat{P}_+^{\boldsymbol{n}} = (a_0 + a_1\, \boldsymbol{m} \cdot \boldsymbol{n})\, \hat{1} + (b_n\, \boldsymbol{n} + b_m\, \boldsymbol{m} + b_\times\, \boldsymbol{m} \times \boldsymbol{n}) \cdot \hat{\boldsymbol{\sigma}}, \tag{2}$$

and find the coefficients a_0, a_1, b_n, b_m and $b_\times$.

(f) **[4 points]** Suppose the system is prepared in the state $|\boldsymbol{n}, +\rangle$, and subsequently the operator $\boldsymbol{m} \cdot \hat{\boldsymbol{\sigma}}$ is measured. Calculate the probability W of finding the system in the state $|\boldsymbol{m}+\rangle$ as a function of the angle θ between $\boldsymbol{n}$ and $\boldsymbol{m}$. Verify your answer in the limiting cases $\boldsymbol{m} = \boldsymbol{n}$ and $\boldsymbol{m} = -\boldsymbol{n}$.

Hint: The probability $W = |\langle \boldsymbol{m}, +|\boldsymbol{n}, +\rangle|^2$ is the square of the inner product. Find how it is related to the trace $\mathrm{Tr}\, \hat{P}_+^{\boldsymbol{m}} \hat{P}_+^{\boldsymbol{n}}$ and then evaluate the trace of Eq. (2).

Pauli matrices: $\hat{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\hat{\sigma}_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $\hat{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Problem A.4

This problem studies a relation between the two characteristics of an ideal Fermi gas of electrons in a metal: the spin susceptibility χ and the specific heat c_V. The Fermi gas has a chemical potential μ, which, in the limit of zero temperature T, is the Fermi energy ε_F.

(a) **[5 points]** The energy density of states $D(\varepsilon) = (dN/d\varepsilon)/V$ is defined as the number of electronic states dN, including both spin orientations, per energy interval $d\varepsilon$ per unit volume V. Calculate $D(\varepsilon)$ for the energy dispersion $\varepsilon = \boldsymbol{p}^2/2m$ in three dimensions (3D), where $\boldsymbol{p}$ is the momentum and m is the mass of an electron. Express the density of states at the Fermi level $D(\varepsilon_F)$ in terms of ε_F and m.

(b) **[8 points]** The entropy of the Fermi gas is (k is the Boltzmann constant)

$$S = kV \int d\varepsilon\, D(\varepsilon)\, s(\varepsilon), \qquad s(\varepsilon) = -f(\varepsilon)\ln f(\varepsilon) - [1 - f(\varepsilon)]\ln[1 - f(\varepsilon)], \tag{1}$$

where $f(\varepsilon) = [e^{(\varepsilon-\mu)/kT} + 1]^{-1}$ is the Fermi distribution function. Sketch the function $s(\varepsilon)$ in the case $kT \ll \varepsilon_F$. Indicate the position and the width of a peak in $s(\varepsilon)$.

Given the shape of $s(\varepsilon)$, argue that Eq. (1) can be approximated as $S \approx VD(\varepsilon_F)\int d\varepsilon\, s(\varepsilon)$ for $kT \ll \varepsilon_F$. Substituting $s(\varepsilon)$ from Eq. (1) and integrating by parts, show that

$$S \approx -\frac{VD(\varepsilon_F)}{T}\int d\varepsilon\, (\varepsilon - \mu)^2\, \frac{df(\varepsilon)}{d\varepsilon}. \tag{2}$$

Using Eq. (3), find the temperature dependence $S(T)$ and the corresponding coefficient.

(c) **[2 points]** Given the formula for the heat $dQ = T\,dS$ at constant volume, the specific heat capacity per unit volume is $c_V = (T/V)\,(\partial S/\partial T)_V$. Using $S(T)$ from Part (b), obtain c_V at $kT \ll \varepsilon_F$ and describe how it depends on T.

(d) **[8 points]** Suppose the degenerate electron gas is subject to a weak magnetic field $\boldsymbol{B}$. The energies for spins parallel and antiparallel to $\boldsymbol{B}$ change by $\mp\mu_B B$, where μ_B is the magnetic moment of an electron. Ignore orbital effects of the magnetic field.

The magnetization of the system is defined as $M = \mu_B(N_\uparrow - N_\downarrow)/V$, where $N_\uparrow$ and $N_\downarrow$ are the numbers of spins parallel and antiparallel to $\boldsymbol{B}$. Obtain a general formula for $M(B)$ as an integral over $d\varepsilon$ in terms of $D(\varepsilon)$ and the Fermi distribution functions $f(\varepsilon \mp \mu_B B)$ for the shifted energies.

Calculate the spin susceptibility $\chi = dM/dB$ in the limit $\mu_B B \ll \varepsilon_F$ and $kT \ll \varepsilon_F$. Express the answer in terms of μ_B and $D(\varepsilon_F)$.

(e) **[2 points]** Calculate the ratio $c_V/(T\chi)$ of the specific heat and the product of the spin susceptibility and temperature. Does this ratio depend on temperature T and the density of states $D(\varepsilon_F)$?

Useful integral

$$I = \int_{-\infty}^{+\infty} dx\, \frac{x^2\, e^x}{(e^x + 1)^2} = \frac{\pi^2}{3}. \tag{3}$$