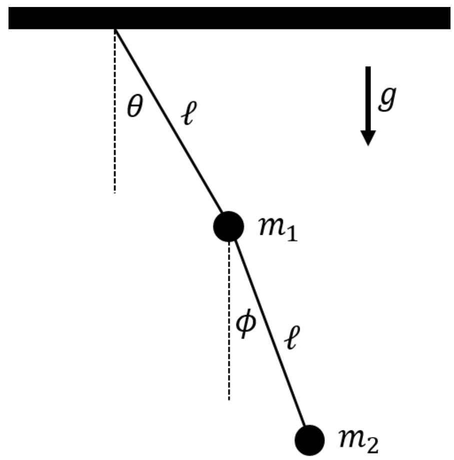
Problem I.1



Consider a double pendulum made with masses m_1 and m_2 hanging from a fixed top support by identical massless rods of length ℓ, as shown. A uniform gravitational acceleration g acts on this system.

(a) [**4 points**] Determine the Lagrangian of the system in terms of the angle coordinates θ and ϕ. (Do not assume that the angles are small. You may use appropriate Cartesian coordinates at first, but then convert to the angle coordinates.) Use $m_T \equiv (m_1 + m_2)$ for convenience.

$\mathcal{L} = T - V$. Using the support point as the origin and measuring x to the right and y downward, the kinetic term for mass 1 can be written as $T_1 = \frac{1}{2}m_1(\dot{x}_1^2 + \dot{y}_1^2)$ with $x_1 = \ell \sin\theta$ and $y_1 = \ell(1 - \cos\theta)$. Taking derivatives and substituting, $T_1 = \frac{1}{2}m_1\ell^2\dot{\theta}^2$.

The kinetic term for mass 2 is a bit more complicated since $x_2 = \ell(\sin\theta + \sin\phi)$ and $y_2 = \ell(2 - \cos\theta - \cos\phi)$. We get
$T_2 = \frac{1}{2}m_2\ell^2\left[(\dot{\theta}\cos\theta + \dot{\phi}\cos\phi)^2 + (\dot{\theta}\sin\theta + \dot{\phi}\sin\phi)^2\right] = \frac{1}{2}m_2\ell^2\left[\dot{\theta}^2 + \dot{\phi}^2 + 2\dot{\theta}\dot{\phi}\cos(\theta - \phi)\right]$,
making use of the trig identity $\cos\theta\cos\phi + \sin\theta\sin\phi = \cos(\theta - \phi)$.

The potential energy term is
$V = -m_1 g y_1 - m_2 g y_2 = g\ell\left[m_1 + 2m_2 - (m_1 + m_2)\cos\theta - m_2\cos\phi\right]$, but because the constant terms have no effect on the dynamics, we can write this more simply as
$V = -g\ell\left[m_T\cos\theta + m_2\cos\phi\right]$.

Putting those things together,
$\mathcal{L} = \frac{1}{2}m_T\ell^2\dot{\theta}^2 + \frac{1}{2}m_2\ell^2\dot{\phi}^2 + m_2\ell^2\dot{\theta}\dot{\phi}\cos(\theta - \phi) + g\ell\left[m_T\cos\theta + m_2\cos\phi\right]$.

(b) [**6 points**] Now, making the approximation that both angles are small, use the Lagrangian to determine coupled equations of motion for the angle coordinates in reasonably simple form.

Keeping terms to second order in the angle coordinates and their derivatives,
$\mathcal{L} \approx \frac{1}{2}m_T\ell^2\dot{\theta}^2 + \frac{1}{2}m_2\ell^2\dot{\phi}^2 + m_2\ell^2\dot{\theta}\dot{\phi} - \frac{g}{2}\ell\left[m_T\theta^2 + m_2\phi^2\right]$.

Evaluating the Lagrange equations:

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = \frac{\partial \mathcal{L}}{\partial \theta} \quad \rightarrow \quad m_T \ell^2 \ddot{\theta} + m_2 \ell^2 \ddot{\phi} = -m_T g \ell \theta \tag{1}$$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \frac{\partial \mathcal{L}}{\partial \phi} \quad \rightarrow \quad m_2 \ell^2 \ddot{\phi} + m_2 \ell^2 \ddot{\theta} = -m_2 g \ell \phi \tag{2}$$

or, simplifying,

$$m_T \ddot{\theta} + m_2 \ddot{\phi} = -m_T \frac{g}{\ell}\theta \tag{3}$$

$$m_2 \ddot{\theta} + m_2 \ddot{\phi} = -m_2 \frac{g}{\ell}\phi \tag{4}$$

(c) **[3 points]** Qualitatively describe the possible normal-mode (periodic) motions of the system.

One mode will have both masses swinging the same direction at a given time, while the other will have them going opposite directions. These could be called "symmetric" and "antisymmetric", although those terms are not really accurate for a double pendulum like they are for a pair of spring-coupled side-by-side pendula.

(d) **[8 points]** Determine the frequencies of the normal modes, still assuming small-angle oscillations.

We require a periodic solution of the form $\theta(t) = A \sin \omega t$, $\phi(t) = B \sin \omega t$. Given the nature of the modes, we can choose the phase so that A and B are both real numbers, though they may have opposite signs. Our pair of equations becomes

$$\omega^2 m_T A + \omega^2 m_2 B = m_T \frac{g}{\ell} A \tag{5}$$

$$\omega^2 m_2 A + \omega^2 m_2 B = m_2 \frac{g}{\ell} B \tag{6}$$

or, dividing out m_2 from the second equation and writing in in matrix form,

$$\begin{bmatrix} \omega^2 m_T - m_T g/\ell & \omega^2 m_2 \\ \omega^2 & \omega^2 - g/\ell \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{7}$$

We find the values of ω which satisfy this by requiring the determinant to equal zero:

$$(\omega^2 m_T - m_T g/\ell)(\omega^2 - g/\ell) - m_2 \omega^4 = 0 \tag{8}$$

$$\rightarrow \quad (\omega^2 - g/\ell)^2 = \frac{m_2}{m_T}\omega^4 \tag{9}$$

$$\rightarrow \quad \omega^2 - g/\ell = \pm\sqrt{m_2/m_T}\,\omega^2 \tag{10}$$

$$\rightarrow \quad \omega^2 = \frac{g/\ell}{1 \pm \sqrt{m_2/m_T}} \tag{11}$$

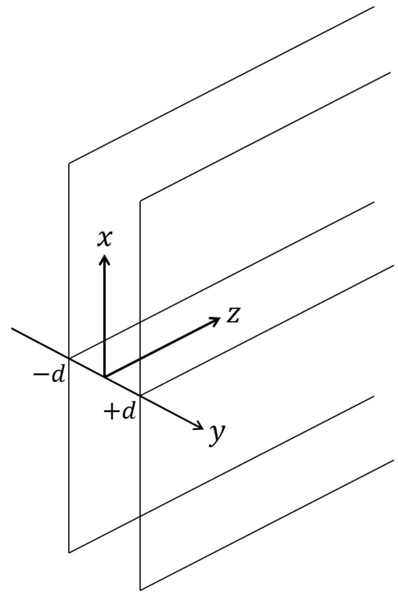
$$\rightarrow \quad \omega = \sqrt{\frac{g/\ell}{1 \pm \sqrt{m_2/m_T}}} \tag{12}$$

One could also express this using $\omega_0 \equiv \sqrt{g/\ell}$.

(e) [**4 points**] Qualitatively, interpreting your part (d) answer, what happens to the normal-mode frequencies if you decrease $m_1 \to 0$ while keeping m_2 fixed?

Decreasing m_1 makes $m_T \to m_2$, so $\sqrt{m_2/m_T} \to 1$. Thus, one of the normal-mode frequencies approaches a constant value, $\omega = \sqrt{g/2\ell}$ (effectively, a single mass on the end of a rod with length 2ℓ), while the other normal-mode frequency increases toward infinity.

Problem I.2



The figure shows a slab of dielectric material which extends to $\pm\infty$ in the x and z directions but extends only from $-d$ to $+d$ in the y direction. In SI units, the slab has dielectric constant $\epsilon > \epsilon_0$ and is nonmagnetic. Air ($\epsilon \approx \epsilon_0$) surrounds the slab. We consider an electromagnetic wave propagating in the z direction through the slab. The Maxwell Equations governing this system are given in SI units as

$$\partial_t \mathbf{B} = -\nabla \times \mathbf{E}, \quad \partial_{\mathbf{t}} \epsilon \mathbf{E} = \nabla \times \mathbf{B}/\mu_{\mathbf{0}}, \quad \textbf{where} \;\; \epsilon_{\mathbf{0}}\mu_{\mathbf{0}} = \mathbf{1}/\mathbf{c^2}. \tag{1}$$

Assume, for this mode, that $B_y = B_z = 0$, and that the other field components (B_x and the electric field vector $\mathbf{E}$) *do not vary with* x. In this problem you are going to show that, with certain constraints, the wave is *guided* in the z direction, in the sense that the fields decrease rapidly with $|y|$ outside the dielectric slab.

(a) **[5 points]** Starting from the Maxwell equations, write down, *in Cartesian coordinates*, the equation for the time evolution of B_x. Assume constant ϵ. Noting that the B_x equation couples to two particular components of $\mathbf{E}$, write down the equations for the time evolution of these coupled $\mathbf{E}$ components, and observe that the resulting system of equations is closed. Combine these equations to obtain a partial differential equation for B_x alone.

To get at the time evolution of B_x, use the $\hat{\mathbf{x}}$ component of Faraday's law: $\partial_t B_x = -\partial_y E_z + \partial_z E_y$.

The electric field components E_y and E_z must be related back to B_x by Ampere's law. Since $\partial_x = 0$, we find $\partial_t \epsilon E_y = \partial_z B_x/\mu_0$, and $\partial_t \epsilon E_z = -\partial_y B_x/\mu_0$. The system of equations for B_x, E_y, and E_z is closed.

Taking the time derivative of the Faraday-law equation, multiplying by ϵ and substituting in the other expressions, we get

$$\epsilon\mu_0 \partial_t^2 B_x = \partial_y^2 B_x + \partial_z^2 B_x \,. \tag{2}$$

(b) [**4 points**] Assume that the wave propagates in the waveguide at a given frequency ω and with a given wavenumber k in the z-direction. Let $B_x \to B_x(y)e^{ikz-i\omega t}$ and thus deduce the ordinary differential equation that must be satisfied by $B_x(y)$, separately inside and outside the slab. Deduce also how each coupled component of $\mathbf{E}(\mathbf{y})$ is related to $B_x(y)$, for given ω and k.

Insert $B_x \to B_x(y)e^{ikz-i\omega t}$ into the differential equation from part (a). That implies $-\epsilon\mu_0\omega^2 B_x = \partial_y^2 B_x - k^2 B_x$, which we can rewrite as $\frac{\partial^2}{\partial y^2}B_x = (k^2 - \epsilon\mu_0\omega^2)B_x$.

Further, since the electric field components must also have $e^{-i\omega t}$ time dependence, the Ampere's law relationships become $-\omega\epsilon\mu_0 E_y = kB_x$ and $-i\omega\epsilon\mu_0 E_z = -\frac{\partial B_x}{\partial y}$.

(c) [**4 points**] From the general Maxwell equations in dielectric media given above, state the boundary ("pillbox") conditions satisfied by $\mathbf{E}$ across the slab boundary $y = d$. Apply these for each of the coupled $\mathbf{E}$ variables involved in our wave and thus deduce the boundary conditions on B_x and its derivative across the discontinuity.

From Faraday's Law, $[E_{tangential}] = 0$. From Coulomb, $[\epsilon E_{normal}] = 0$. Thus, $[\epsilon E_y] = 0$, and $[E_z] = 0$. Using results from (b), these yield $[B_x] = 0$ and $[(1/\epsilon)(d/dy)B_x] = 0$.

(d) [**4 points**] Up to a constant, write down a solution for $B_x(y)$ outside the slab. For a guided wave, the solution must decay exponentially to zero as y becomes large. What condition does this place on k and ω?

Outside, $\epsilon = \epsilon_0$. So, $B_x = Be^{-\beta(y-d)}$, where $\beta = (k^2 - \epsilon_0\mu_0\omega^2)^{1/2}$ and must have $k^2 > \epsilon_0\mu_0\omega^2$. Since $\epsilon_0\mu_0 = 1/c^2$, those can also be written as $\beta = (k^2 - \omega^2/c^2)^{1/2}$ and as $k^2 > \omega^2/c^2$ or $|k| > \omega/c$.

(e) [**4 points**] Assuming even solutions about $y = 0$, write down, up to a constant, a solution for $B_x(y)$ inside the slab. What condition does this place on k, ω, and ϵ?

Inside, $\epsilon > \epsilon_0$. To get an even solution, we must have a wave equation with solution $B_x(y) = A\cos(\alpha y)$, where $\alpha = (\epsilon\mu_0\omega^2 - k^2)^{1/2}$ and must have $k^2 < \epsilon\mu_0\omega^2$.

(f) [**4 points**] Apply the boundary conditions at $z = d$ obtained in (c) and, so, find a dispersion relation of the form $(\epsilon/\epsilon_0)\beta = \alpha\tan(\alpha d)$, where α and β are functions of ω and k. Specify both these functions.

We now apply the boundary conditions on B_x and its derivative, as resulted from part (c). These give $B = A\cos(\alpha d)$, and $(\epsilon/\epsilon_0)\beta B = A\alpha\sin(\alpha d)$. Eliminate A and B to get the required dispersion relation. Here, $\alpha \equiv \sqrt{\epsilon\mu_0\omega^2 - k^2}$ and $\beta \equiv \sqrt{k^2 - \epsilon_0\mu_0\omega^2} = \sqrt{k^2 - \omega^2/c^2}$.

Problem I.3

A small particle bound to a surface defect by a centrally attractive force may act as a two-dimensional classical harmonic oscillator. Assume that this particle is in thermal equilibrium with its environment, which has temperature T.
(Note that the parts of this problem are not all sequential; if you get stuck on one part, you may still be able to do some of the later parts.)

(a) **[3 points]** Suppose the restoring force on the particle produces a natural (angular) frequency ω for linear oscillations in either the x direction or the y direction. Write down (you don't need to derive it) the Hamiltonian of the particle as a function of its mass m, instantaneous position (x, y), and momentum $\mathbf{p}$.

The kinetic term is $p^2/2m$, while the potential terms are $\frac{1}{2}m\omega^2x^2$ in the x direction and similar in the y direction. Thus the Hamiltonian is $\mathcal{H} = \frac{p^2}{2m} + \frac{1}{2}m\omega^2(x^2 + y^2)$. (Here, p is the magnitude of the momentum vector $\mathbf{p}$, which has components p_x and p_y.)

(b) **[6 points]** Calculate the partition function for this system, doing integrals where appropriate to reduce it to a simple form involving T. (Recall that the partition function involves Planck's constant even for classical systems, as a conventional unit of phase space.)

The quadratic form of this 2D oscillator is equivalent to two independent 1D oscillators, so to calculate the partition function, we can either integrate over the four parameters (x, p_x, y, p_y), or just calculate the partition function for a 1D oscillator and square it. Taking the latter approach and using $\beta \equiv \frac{1}{k_B T}$,

$$
\begin{aligned}
Z_{1\mathrm{D}} &= \frac{1}{h}\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dp_x\, e^{-\beta E} && (1)\\
&= \frac{1}{h}\int_{-\infty}^{\infty} dx e^{\frac{\beta}{2}m\omega^2x^2} \int_{-\infty}^{\infty} dp_x\, e^{-\frac{\beta}{2m}p_x^2} && (2)\\
&= \frac{1}{h}\sqrt{\frac{\pi}{\frac{\beta}{2}m\omega^2}}\sqrt{\frac{\pi}{\beta/2m}} && (3)\\
&= \frac{2\pi}{h\beta\omega} && (4)\\
&= \frac{k_B T}{\hbar\omega} && (5)
\end{aligned}
$$

Then the partition function of the 2D particle system is

$$
Z = (Z_{1\mathrm{D}})^2 = \left(\frac{k_B T}{\hbar\omega}\right)^2. \qquad (6)
$$

If you don't recall why the partition function of a classical system involves Planck's constant, see https://physics.stackexchange.com/questions/184947/why-is-the-partition-function-divided-by-h3n-n , for instance.

(c) **[5 points]** Now consider an ensemble of N of these two-dimensional harmonic oscillators, taking them to be non-interacting and distinguishable. Find the Helmholtz free energy and the entropy of the ensemble in terms of T, ω, and constants.

The partition function for N such particles is

$$Z = \left(\frac{k_B T}{\hbar\omega}\right)^{2N} . \tag{7}$$

The Helmholtz free energy is

$$F = -k_B T \,\ln Z = -N k_B T \,\ln\left(k_B T/\hbar\omega\right) \tag{8}$$

and the entropy is

$$\begin{aligned} S = \frac{-\partial F}{\partial T} &= N k_B \ln\left(k_B T/\hbar\omega\right) + N k_B T \left(\frac{\hbar\omega}{k_B T}\right)\left(\frac{k_B}{\hbar\omega}\right) && (9) \\ &= N k_B \ln\left(k_B T/\hbar\omega\right) + N k_B && (10) \\ &= N k_B \left[\ln\left(k_B T/\hbar\omega\right) + 1\right] && (11) \end{aligned}$$

(d) **[2 points]** Now go back to focusing on just a *single* bound classical particle. While it is in thermal equilibrium with its environment, its energy will *not* be constant over time. Explain why in one or two sentences.

There will be random exchange of energy with its environment, so the energy of the particle will fluctuate. Only its *average* energy is directly determined by the temperature.

(e) **[4 points]** Use the partition function from part (b) to calculate the average energy of this (single) bound-particle system expressed in terms of T. (Or, for partial credit, determine its average energy in some other way.)

The ensemble average is given by

$$\begin{aligned} \overline{E} &= \frac{\int\int E e^{-\beta E}}{\int\int e^{-\beta E}} = \frac{-\partial Z/\partial\beta}{Z} && (12) \\ &= \frac{\frac{-\partial}{\partial\beta}\,\frac{1}{(\hbar\omega)^2\beta^2}}{\frac{1}{(\hbar\omega)^2\beta^2}} && (13) \\ &= 2\,\beta^{-1} = 2k_B T\,. && (14) \end{aligned}$$

Note that this can also be written as

$$\begin{aligned} \overline{E} &= \frac{-\partial \ln Z}{\partial\beta} && (15) \\ &= \frac{-\partial}{\partial\beta}\,\ln\left(\left(\frac{1}{\hbar\omega\beta}\right)^2\right) = \frac{-\partial}{\partial\beta}\,\left[-2\ln\hbar\omega - 2\ln\beta\right] && (16) \\ &= 2/\beta = 2k_B T\,. && (17) \end{aligned}$$

Either way, that agrees with what we expect from counting degrees of freedom: $\frac{1}{2}k_B T$ for each of four degrees of freedom. If they don't use the partition function but just argue based on degrees of freedom and get the right answer that way, give 2 points.

Alternatively, instead of using either of the handy partition-function derivatives above, it should be possible to calculate the integral $\int\int E e^{-\beta E}$ directly using standard definite integrals and get the answer for $\overline{E}$ using that. Give full credit if they slog through that and get the right answer.

(f) [**5 points**] Calculate the root-mean-square fluctuation in the energy of this system, expressed in terms of T.

We need to find $\overline{(\Delta E)^2} = \overline{E^2} - \overline{E}^2$ and then take the square root of that.

$$\overline{E^2} = \frac{\int\int E^2 e^{-\beta E}}{\int\int e^{-\beta E}} = \frac{\frac{\partial^2 Z}{\partial \beta^2}}{Z} \tag{18}$$

$$= \frac{\frac{\partial^2}{\partial \beta^2}\frac{1}{(\hbar\omega)^2\beta^2}}{\frac{1}{(\hbar\omega)^2\beta^2}} \tag{19}$$

$$= 6\,\beta^{-2} = 6(k_B T)^2\,. \tag{20}$$

So

$$\Delta E_{\text{rms}} = \sqrt{6(k_B T)^2 - (2k_B T)^2} = \sqrt{2}\,k_B T\,. \tag{21}$$

Possibly useful:

$$\int_{-\infty}^{\infty} e^{-ax^2}\,dx = \sqrt{\frac{\pi}{a}} \tag{22}$$

$$\int_{0}^{\infty} x e^{-ax^2}\,dx = \frac{1}{2a} \tag{23}$$

$$\int_{-\infty}^{\infty} x^2 e^{-ax^2}\,dx = \frac{1}{2a}\sqrt{\frac{\pi}{a}} \tag{24}$$

$$\int_{0}^{\infty} x^3 e^{-ax^2}\,dx = \frac{1}{2a^2} \tag{25}$$

$$\int_{-\infty}^{\infty} x^4 e^{-ax^2}\,dx = \frac{3}{4a^2}\sqrt{\frac{\pi}{a}} \tag{26}$$

Problem I.4

The $W^{\pm}$ bosons were first discovered in the collisions of beams of protons and antiprotons. The two beams circulate in opposite directions in a large ring and have the same energy, $E_{\rm b}$, which is much greater than the proton rest energy. The reaction is understood to be a collision between a **u** quark from the proton and an anti-**d** quark from the antiproton. The quarks can carry any fraction of the beam energy, have essentially zero mass, and essentially zero momentum transverse to the beam axis.

(a) [**5 points**] Let x_1 be the fraction of the proton's momentum carried by the **u** quark, and let x_2 be the fraction of the antiproton's momentum carried by the anti-**d** quark. (Both x_1 and x_2 are between 0 and 1.) Determine the necessary relationship between x_1 and x_2 such that they annihilate, producing a W particle (with mass $M_{\rm W}$) and nothing else. This relationship should be in terms of $M_{\rm W}$ and the beam energy.

4-momentum vectors simplify to 2-component vectors, (p_z, E), if the transverse momenta are negligible. Setting $c = 1$ and taking the quarks to be massless, the 2-momenta of the **u** and anti-**d** quarks in the lab frame are $(x_1 E_{\rm b}, x_1 E_{\rm b})$ and $(-x_2 E_{\rm b}, x_2 E_{\rm b})$. Those must add to give the 2-mometum of the W particle, $(p_{\rm W}, E_{\rm W})$, such that the invariant mass is the W mass:

$$M_{\rm W}^2 = E_{\rm W}^2 - p_{\rm W}^2 = (x_1 + x_2)^2 E_{\rm b}^2 - (x_1 - x_2)^2 E_{\rm b}^2 = 4 x_1 x_2 E_{\rm b}^2 \,. \tag{1}$$

Thus, we require

$$x_1 x_2 = \frac{M_{\rm W}^2}{4E_{\rm b}^2} = \left(\frac{M_{\rm W}}{2E_{\rm b}}\right)^2 \tag{2}$$

(b) [**5 points**] Based on your answer to (a), what is the permissible range of values for the W particle's momentum component parallel to the proton beam axis? That is, calculate $p_{\rm max}$.

Its momentum component is $(x_1 - x_2)E_{\rm b}$. This is maximized by making x_1 large and x_2 small, but of course x_1 can't be greater than 1. From the condition we found in part (a), that requires $x_2 = \left(\frac{M_{\rm W}}{2E_{\rm b}}\right)^2$. So then the W's momentum component is

$$\begin{aligned} p_{\rm max} &= \left(1 - \frac{M_{\rm W}^2}{4E_{\rm b}^2}\right) E_{\rm b} && (3) \\ &= E_{\rm b} - \frac{M_{\rm W}^2}{4E_{\rm b}} && (4) \end{aligned}$$

or, putting factors of c back in,

$$p_{\rm max} = \frac{E_{\rm b}}{c} - \frac{M_{\rm W}^2 c^3}{4E_{\rm b}} \,. \tag{5}$$

The allowed range (symmetric around zero) is from $-p_{\rm max}$ to $p_{\rm max}$, but that symmetry is pretty obvious, so it's OK to just give $p_{\rm max}$.

(c) **[5 points]** A variable commonly used to characterize a particle emerging from a beam-beam collision is the "rapidity",

$$\eta \equiv \frac{1}{2} \ln \left(\frac{E + p_L}{E - p_L} \right)$$

where p_L is the component of the particle's momentum parallel to the proton beam axis and E is the energy of the particle. What is the maximum value of η possible for a W particle produced in the collision described above (in the lab frame)?

(In this scenario, its momentum is much larger than its mass, but M_{W} cannot be totally neglected. Expand to lowest order and simplify to get a finite (approximate) value for the maximum η in terms of M_{W} and $p_{\max}$.)

The W produced in this reaction has longitudinal momentum but no transverse momentum in the lab frame.

$$\begin{aligned} E = \sqrt{p_L^2 + M_{\mathrm{W}}^2} &= p_L \left(1 + \frac{M_{\mathrm{W}}^2}{p_L^2} \right)^{1/2} && (6) \\ &\approx p_L \left(1 + \frac{M_{\mathrm{W}}^2}{2p_L^2} \right) . && (7) \end{aligned}$$

Inserting that into the rapidity definition, dividing through by p_L and dropping the term beyond lowest order in the numerator,

$$\begin{aligned} \eta_{\max} &= \frac{1}{2} \ln \left(\frac{2}{M_{\mathrm{W}}^2/2p_L^2} \right) && (8) \\ &= \frac{1}{2} \ln \left(\frac{4p_{\max}^2}{M_{\mathrm{W}}^2} \right) && (9) \\ &= \ln \left(\frac{2p_{\max}}{M_{\mathrm{W}}} \right) && (10) \end{aligned}$$

(d) **[5 points]** The collision will actually produce multiple particles, and the W will decay almost instantly to other particles. Rapidity is a frame-dependent quantity (which is why we specified the lab frame in the previous part). However, when two particles emerge from the same collision, the rapidity *difference* $\Delta\eta \equiv \eta_1 - \eta_2$ is invariant under Lorentz transformations along the beam axis. Prove that explicitly.

Dropping the L subscripts, the rapidity difference is

$$\begin{aligned} \Delta\eta &= \frac{1}{2} \ln \left(\frac{E_1 + p_1}{E_1 - p_1} \right) - \frac{1}{2} \ln \left(\frac{E_2 + p_2}{E_2 - p_2} \right) && (11) \\ &= \frac{1}{2} \ln \left(\frac{(E_1 + p_1)(E_2 - p_2)}{(E_1 - p_1)(E_2 + p_2)} \right) . && (12) \end{aligned}$$

The Lorentz transformation (boost) in the z direction takes $p \to \gamma(p - \beta E)$ and $E \to \gamma(E - \beta p)$. Applying those, in the new frame

$$\Delta\eta = \frac{1}{2}\ln\left(\frac{\gamma((E_1 - \beta p_1) + (p_1 - \beta E_1))\,\gamma((E_2 - \beta p_2) - (p_2 - \beta E_2))}{\gamma((E_1 - \beta p_1) - (p_1 - \beta E_1))\,\gamma((E_2 - \beta p_2) + (p_2 - \beta E_2))}\right) \tag{13}$$

$$= \frac{1}{2}\ln\left(\frac{(E_1 + p_1)(1-\beta)\,(E_2 - p_2)(1+\beta)}{(E_1 - p_1)(1+\beta)\,(E_2 + p_2)(1-\beta)}\right) \tag{14}$$

$$= \frac{1}{2}\ln\left(\frac{(E_1 + p_1)\,(E_2 - p_2)}{(E_1 - p_1)\,(E_2 + p_2)}\right) \tag{15}$$

which is the same as in the other frame.

(e) [**5 points**] Many collider detectors installed at proton-antiproton collision points have a solenoidal magnet centered on the interaction point and co-axial with the beam axis, and either a silicon or wire-chamber tracking detector to record the paths of charged particles emerging from the collision. Explain how the path of a charged particle in this region is used to determine the p_L and E quantities used to calculate its rapidity (ignoring, in this case, any information that may be available from a calorimeter). (Hint: consider the direction of the magnetic field produced by a solenoid.)

The magnetic field produced by the solenoid points along the beam axis, so a particle's velocity component parallel to the beam axis does not produce any Lorentz force; only the transverse velocity component produces a Lorentz force that curves the particle's path. Thus, the curvature of the path directly measures the transverse momentum, and that together with the angle of the path in 3D determines the longitudinal momentum component, p_L. The total momentum vector magnitude, combined with the mass of the particle (if not negligible), can be used to calculate E. In general, you need to either know or assume the type of particle (electron, muon, pion, or whatever) so that you can use the correct mass to calculate E from $\mathbf{p}$.

Problem I.5

A uniform string of length L under tension τ undergoes small transverse oscillations. The mass per unit length of the string is given by μ, and the equilibrium position of the string lies along the x axis. The transverse displacement of the string at the point with coordinate x at time t is denoted by $y(x, t)$. One end of the string at $x = 0$ is attached to a fixed support so that the transverse displacement at this point vanishes, $y(0, t) = 0$. The other end of the string is attached to a point particle of mass m that is restricted to lie along the line $x = L$, but is free to move without friction along the y direction.

(a) [**4 points**] Write down the wave equation of motion for small amplitude displacements $y(x, t)$. Express the velocity of propagation of transverse waves in terms of τ and μ.

The wave equation in one dimension takes the form

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2} \tag{1}$$

For a vibrating string the wave velocity v is given by $v = \sqrt{\tau/\mu}$.

(b) [**5 points**] By applying Newton's 2nd Law to the mass m, show that the appropriate boundary condition for *small* displacements along y at $x = L$ has the form

$$\kappa\frac{\partial y}{\partial x} = -\frac{\partial^2 y}{\partial t^2}\ . \tag{2}$$

Express the constant κ in terms of the physical parameters in the problem.

Since the mass can only move along the y direction, we apply Newton's Law along this direction. The component of the tension τ pulling on the mass along the y direction is given by $(-\tau\sin\theta)$. Then at $x = L$ we have

$$m\frac{d^2 y}{dt^2} = -\tau\sin\theta \tag{3}$$

But for small dispacements of the string,

$$\sin\theta \approx \tan\theta = \frac{dy}{dx} \tag{4}$$

Then at $y = L$ we have the boundary condition

$$m\frac{d^2 y}{dt^2} = -\tau\frac{dy}{dx} \tag{5}$$

Equivalently,

$$-\frac{d^2 y}{dt^2} = \kappa\frac{dy}{dx} \tag{6}$$

with $\kappa = \tau/m$.

(c) [**10 points**] Use the boundary condition above to obtain a transcendental equation that implicitly determines the characteristic frequencies of the normal modes of this system. (You may write the equation in terms of a wavenumber k instead of a frequency parameter).

We look for solutions of the form

$$y(x,t) = X(x)T(t) \tag{7}$$

Then

$$\frac{1}{X}\frac{d^2X}{dx^2} = \frac{1}{v^2}\frac{1}{T}\frac{d^2T}{dt^2} \tag{8}$$

Since the left hand side is a function of x alone and the right hand side a function of t alone, both sides must equal a constant which we denote by $-k^2$. Then,

$$\frac{d^2T}{dt^2} + k^2v^2T = 0 \tag{9}$$

This has solution $T(t) = C_1 \sin(kvt + \phi)$, where C_1 is a constant and ϕ is a constant phase. Also,

$$\frac{d^2X}{dx^2} + k^2X = 0 \tag{10}$$

with solution

$$X(x) = D_1 \sin(kx) + D_2 \cos(kx) \tag{11}$$

From the boundary condition $X = 0$ at $x = 0$ we have $D_2 = 0$, so $X(x) = D_1 \sin kx$. Then

$$y(x,t) = C \sin(kx) \sin(kvt + \phi) \tag{12}$$

Imposing the boundary condition at $x = L$ we obtain

$$k^2v^2 \sin(kL) = k\kappa \cos(kL) \tag{13}$$

Recalling that $\kappa = \tau/m = v^2(\mu/m)$, we obtain the condition

$$\tan(kL) = \frac{\mu}{mk}. \tag{14}$$

This transcendental equation implicitly determines the frequencies of the normal modes.

Note: If you can't get the answer to this part, you can still answer parts (d) and (e) through other lines of reasoning for partial credit.

(d) [**3 points**] Use this transcendental equation to obtain the solution for the wavelengths of the normal modes in the limit that $m \to \infty$, (or, more precisely, $m \gg \mu L$). Give a physical interpretation of your result.

We write the transcendental equation as

$$kL \tan(kL) = \frac{\mu L}{m} \tag{15}$$

In the limit $m \gg \mu L$, the solutions of this equation are approximately given by the roots of $\tan(kL) = 0$. Then

$$k_n = \frac{n\pi}{L} \qquad n \geq 1 \tag{16}$$

Since $k_n = 2\pi/\lambda_n$, we obtain the wavelengths of the normal modes as $\lambda_n = 2L/n$.

The physical explanation is that in the limit that $m \to \infty$, the inertia of the mass is very large and it cannot be displaced from its equilibrium position. This corresponds to having a stiff boundary at $y = L$ so that $y(L) = 0$. The standing waves for a string of length L fixed at both ends have wavelengths $\lambda_n = 2L/n$, which is exactly the result we find.

(e) [**3 points**] Use the equation from part (c) to obtain the solution for the wavelengths of the normal modes in the limit that $m \to 0$. Give a physical interpretation of your result.

In the limit $m = 0$, the normal modes satisfy $\tan(kL) = \infty$. Then

$$k_n = (2n-1)\frac{\pi}{2L} \qquad n \geq 1 \tag{17}$$

The corresponding wavelength is given by

$$\lambda_n = \frac{4L}{2n-1} \qquad n \geq 1 \tag{18}$$

The physical explanation is that in the limit $m = 0$, the end at $x = L$ is free and corresponds to an anti-node.