UNIVERSITY OF MARYLAND
Department of Physics
College Park, Maryland

PHYSICS Ph.D. QUALIFYING EXAMINATION
PART I

September 2, 2020 11:00 a.m. – 3:00 p.m.

Do any four problems. Each problem is worth 25 points.
Start each problem on a new sheet of paper (because different faculty members will be grading each problem in parallel).

Be sure to write your Qualifier ID ("control number") at the top of each sheet — not your name! — and turn in solutions to four problems only. (If five solutions are turned in, we will only grade # 1 - # 4.)

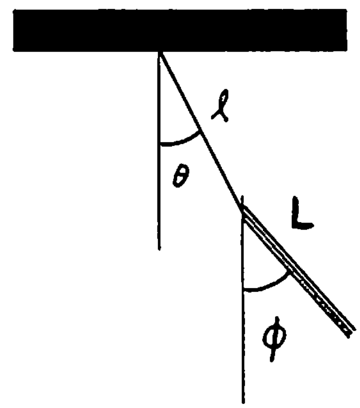
At the end of the exam, when you are turning in your papers, please fill in a "no answer" placeholder form for the problem that you skipped, so that the grader for that problem will have something from every student.

You may keep this packet with the questions after the exam.

Problem I.1

One end of a uniform thin rod, of length L and mass M, is attached to the end of a string, of length ℓ and negligibly small mass, whose other end is fastened to the ceiling. The figure shows that this system at a particular instant while it is oscillating in a plane after being slightly displaced from equilibrium. The configuration of the system at any time can be described by the angle θ of the string from the vertical and the angle ϕ of the rod from the vertical. Friction forces are negligible, and the magnitude of the gravitational acceleration is g.

(a) [**7 points**] Find the Lagrangian of this system in terms of the angles θ and ϕ and their time derivatives. Do **not** yet make any small-angle approximations.
Reminder: The moment of inertia of a uniform thin rod of mass M and length L about an axis perpendicular to the rod through the center of the rod, is $(1/12)ML^2$.



(b) [**6 points**] Now make the small-angle approximation, which we will use hereafter, and derive the equations of motion for the [now small] angles θ and ϕ and their time derivatives.

(c) [**8 points**] Consider the special case in which $L = (3/2)\ell$. What then are the frequencies of the normal modes of oscillation of this system? Express these frequencies in terms of g and ℓ only.

(d) [**4 points**] Which of these frequencies corresponds to that of the normal mode illustrated qualitatively by the figure? Explain briefly the reasoning behind your choice.

Problem I.2

Using cylindrical coordinates, where r is the radial coordinate, ϕ is the azimuthal coordinate, and z is the axial coordinate, consider a cylindrical vacuum-filled cavity of radius a and length ℓ, bounded by perfectly conducting walls at $z = 0$, $z = \ell$, and $r = a$. We seek *time-varying* solutions for the electric field $\vec{E}$ and the magnetic flux density $\vec{B}$ within the cavity ($r \leq a, 0 \leq z \leq \ell$) with the property that they do not depend on either z or ϕ.

(a) [**5 points**] Show that such solutions must have a $\vec{B}$-field that is purely in the ϕ-direction ($\vec{B} = B_\phi(r,t)\hat{\phi}$, where $\hat{\phi}$ is a unit vector in the ϕ-direction), and an $\vec{E}$-field that is purely in the z direction ($\vec{E} = E_z(r,t)\hat{z}$, where $\hat{z}$ is a unit vector in the z-direction).

(b) [**5 points**] Find a wave equation satisfied by $E_z(r,t)$.

(c) [**6 points**] Now assume that the time dependence of the fields is sinusoidal and proportional to $e^{-i\omega t}$. Find a differential equation for the spatial dependence of E_z; that is, taking $E_z(r,t) = e^{-i\omega t}\,\tilde{E}_z(r)$, find a differential equation for $\tilde{E}_z(r)$. Specify the boundary condition on $\tilde{E}_z(r)$.

(d) [**6 points**] Invoking the boundary conditions, give the possible solutions for $\tilde{E}_z$(r); obtain and *discuss* a condition satisfied by the possible frequencies of oscillation ω (see the "Given Information" for a reminder regarding Bessel functions).

(e) [**3 points**] Sketch plots of $\tilde{E}_z(r)$ and $\tilde{B}_\phi(r)$ vs. r for the solution with the second lowest non-zero frequency (where $B_\phi(r,t) = e^{-i\omega t}\,\tilde{B}_\phi(r)$).

Given Information

In cylindrical coordinates,

$$\nabla \times \vec{V} = \left[\frac{1}{r}\frac{\partial V_z}{\partial \phi} - \frac{\partial V_\phi}{\partial z}\right]\hat{r} + \left[\frac{\partial V_r}{\partial z} - \frac{\partial V_z}{\partial r}\right]\hat{\phi} + \frac{1}{r}\left[\frac{\partial}{\partial r}(rV_\phi) - \frac{\partial V_r}{\partial \phi}\right]\hat{z}\,. \tag{1}$$

Bessel's equation of integer order n ($n = 0, 1, 2, \ldots$) is

$$\frac{d^2 f(u)}{d\,u^2} + \frac{1}{u}\frac{df(u)}{du} + \left(1 - \frac{n^2}{u^2}\right) f(u) = 0, \tag{2}$$

which has two independent solutions conventionally denoted $J_n(u)$ and $Y_n(u)$, where J_n and Y_n are called the n^{th}-order Bessel functions, having $J_0(0) = 1$, $J_n(0) = 0$ for $n \geq 1$, and $Y_n(0) = \infty$.

Problem I.3

First consider a classical ideal gas at temperature T consisting of N molecules and initially confined in a volume V_i. Then the gas is allowed to expand to a final volume V_f in two different ways:

(a) [**5 points**] *Free expansion.* The gas is thermally insulated from its environment and experiences free irreversible expansion into a vacuum. Calculate the entropy change of the gas $\Delta S^{gas}_{irr} = S_f - S_i$ by comparing the number of accessible states before and after the expansion.

(b) [**5 points**] *Isothermal expansion.* The gas is in thermal contact with a reservoir of temperature T and experiences a slow reversible quasistatic expansion, e.g. produced by a slow-motion piston that limits the gas volume. Calculate the work W done on the gas in this process, the change $\Delta U = U_f - U_i$ of the internal energy of the gas, and the heat Q transferred to the gas from the environment. Calculate the entropy change of the gas $\Delta S^{gas}_{rev} = S_f - S_i$ in this reversible process by using the formula $\Delta S = Q/T$. Compare your answers for S^{gas}_{irr} and ΔS^{gas}_{rev}. Are the two results the same or different? Explain why.

(c) [**5 points**] What are the entropy changes in the environment for these two cases: ΔS^{env}_{irr} and ΔS^{env}_{rev}? What are the total entropy changes in the gas and the environment for these two cases: $\Delta S^{tot}_{irr} = \Delta S^{gas}_{irr} + \Delta S^{env}_{irr}$ and $\Delta S^{tot}_{rev} = \Delta S^{gas}_{rev} + \Delta S^{rev}_{irr}$? Are ΔS^{tot}_{irr} and ΔS^{tot}_{rev} the same or different? Explain why.

(d) [**5 points**] Now consider a non-interacting degenerate Fermi gas made of N spin-1/2 fermions each of mass m and initially confined in a volume V_i at zero temperature $T = 0$. Calculate the Fermi momentum p_F, the Fermi energy E_F, and the energy per particle U/N. Express U/N in terms of E_F.

(e) [**5 points**] This Fermi gas is thermally insulated from its environment and experiences free irreversible expansion into a vacuum to a final volume V_f. Assume that V_f is sufficiently large so that the Fermi gas becomes non-degenerate, i.e. classical, and is ideal. Calculate the final temperature T_f of the gas after the expansion. Express T_f in terms of E_F obtained above.

Problem I.4

According to recent models of strong interactions based on quantum chromodynamics, a strongly interacting particle and its antiparticle may combine to create an object consisting primarily of gluons and referred to as a *glueball (G)*. One model predicts a glueball with rest mass $M = 1.90\ \mathrm{GeV}/c^2$. Take the proton rest mass to be $940\ \mathrm{MeV}/c^2$. If you choose to do your derivations in units with $c = 1$ in this problem, you should include c in the units of any numerical answer (e.g., GeV/c^2 for a mass.

SPECIAL NOTE: ***Since using smart phones is not allowed and distributing regular calculators is not feasible these days, please do the numerical problems by 1) finding the equation for the solution using variables, 2) rewriting that equation with the appropriate numbers. Include the units of all these numbers and that of the answer. 3) If you could calculate the answer and write it in scientific notation, what would be the power of 10?***

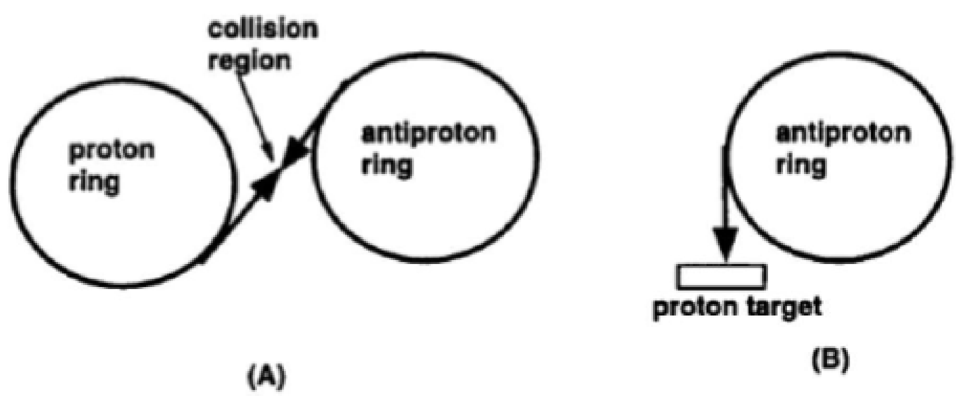


Figure 1: Accelerator Modes

(a) Suppose you wanted to propose an experiment at a $p\bar{p}$ collider to detect the reaction $p + \bar{p} \to G$ (proton plus antiproton combine to create a glueball). In this collider protons with a sharply defined energy are circulated in one ring and anti-protons of the same energy in another. They are then extracted in a collision region in which they approach each other head-on with equal energies from opposite directions. See Fig. A above.

[10 points] At what proton momentum would you look for this reaction to occur? At what total energy?

(b) Another way to make this reaction take place is to use a single ring accelerator and have accelerated particles hit a stationary target (see Fig. B above). Suppose you accelerate antiprotons and have them hit a stationary target of ionized hydrogen.

[15 points] At what anti-proton momentum would you look for the reaction to occur? At what total energy?

Problem I.5

Wave propagation involves the interplay of a restoring force and the inertia of the medium. For waves on the surface of water, the external restoring force comes from gravity, while the inertia arises from the mass density of the water. If the wavelength is longer than a few centimeters, the only significant internal forces come from pressure gradients, since the effects of surface tension and viscosity are negligible. For small amplitudes these waves satisfy a linear equation. In this problem you will deduce their dispersion relation.

The flow in these waves is irrotational ($\nabla \times \mathbf{v} = 0$), so the velocity field $\mathbf{v}$ is the gradient of a scalar velocity potential Φ, i.e. $\mathbf{v} = \nabla\Phi$. Let (x, y, z) be Cartesian coordinates, with y corresponding to the vertical position, $y = 0$ being the undisturbed level of the water surface. Consider in this problem a wave with velocity potential $\Phi(x, y, t)$, propagating in the $\hat{x}$ direction with wavenumber k, and with angular frequency ω. The velocity of the fluid will depend on the depth y, and the wave amplitude is independent of z.

(a) **[6 points]** If the depth h of the body of water is sufficiently large compared to the wavelength, then the depth plays no role in the wave properties. Such waves are called "deep water waves," and the only parameters of the system that might enter the dispersion relation $\omega(k)$ are the acceleration of gravity g, the mass density of water ρ. Using dimensional analysis, determine the form of the dispersion relation for deep water waves, up to an unknown dimensionless constant. If you find that g or ρ does not enter the dispersion relation, give a physical explanation why not.

(b) **[3 points]** Assuming the y dependence factors out, as $\Phi(x, y, t) = G(y)F(x, t)$, write out $F(x, t)$ for the given wave.

(c) **[4 points]** Water is nearly incompressible. Explain why this implies that the velocity potential satisfies Laplace's equation, $\nabla^2\Phi = 0$.

(d) **[6 points]** Use the fact that Φ satisfies Laplace's equation to determine the form of the function $G(y)$ for the case of deep water waves. Note that the fluid velocity must vanish at great depths.

(e) **[3 points]** For small amplitude waves and constant atmospheric pressure on the surface of the water, the velocity potential on the surface satisfies the equation $\partial_t^2\Phi = -g\partial_y\Phi$. Use this surface boundary condition, together with your previous results, to determine the dispersion relation for deep water waves. Compare this dispersion relation with the one you obtained using dimensional analysis in part (a).

(f) **[3 points]** Suppose now that the body of water has a constant depth h that is not effectively infinite. Determine the function $G(y)$, using the appropriate boundary condition at the bottom. (This can be used to find how the dispersion relation depends on h, but you are not asked to do so.)