UNIVERSITY OF MARYLAND
Department of Physics
College Park, Maryland

PHYSICS Ph.D. QUALIFYING EXAMINATION
PART B

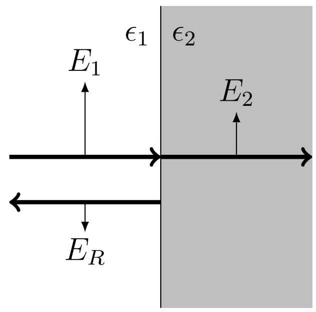
January 15, 2025	**10:00 am – 12:00 pm**
January 16, 2025	**10:00 am – 12:00 pm**

Problem B.1

Consider propagation of electromagnetic waves through non-magnetic, dielectric media.

(a) [**5 points**] Formulate the boundary conditions for the normal and tangential components of the electric fields $\boldsymbol{E}$ and magnetic fields $\boldsymbol{B}$ at the interface between two semi-infinite non-magnetic, dielectric materials with dielectric constants ϵ_1 and ϵ_2.

(b) [**6 points**] An electromagnetic wave of amplitude E_1 propagates normal to the interface, resulting in reflected and transmitted waves of amplitudes E_R and E_2 as shown.

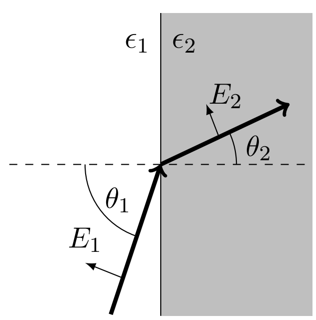


What is the magnetic field amplitude and direction $\boldsymbol{B}_1$ of the incoming wave?

Calculate the electric and magnetic amplitudes and directions of the transmitted and reflected waves: $\boldsymbol{E}_2$, $\boldsymbol{B}_2$, and $\boldsymbol{E}_R$, $\boldsymbol{B}_R$ in terms of E_1.

(c) [**3 points**] Calculate the time-averaged Poynting vectors $\boldsymbol{S}$ for the incoming, reflected, and transmitted waves in Part (b). Compare the sum of the first two with the last one, and formulate a condition that they satisfy.

(d) [**4 points**] An electromagnetic wave propagates at an angle θ_1 with respect to the normal of the interface, with the polarization of $\boldsymbol{E}_1$ in the plane of incidence as drawn below. The transmitted wave $\boldsymbol{E}_2$ propagates on the other side at angle θ_2. We want to determine the Brewster angle θ_1 where any reflected wave vanishes, so that $\boldsymbol{E}_R = 0$.



Given that a reflected wave is produced by radiation from dipoles in medium 2, derive a relation between the angles θ_1 and θ_2 leading to extinction of the reflected wave.

(e) [**7 points**] Using your results from Part (a) and the relationship from Part (d), determine the angles θ_1 and θ_2 such that $\boldsymbol{E}_R = 0$ in terms of ϵ_1 and ϵ_2.

Even if you fail in Part (d), the answer can be obtained from Part (a) with $\boldsymbol{E}_R = 0$.

Problem B.2

Consider an infinitely long straight wire of negligible cross-sectional area with a uniform linear charge density λ at rest in the inertial reference frame K.

(a) [**4 points**] Find the electric $\boldsymbol{E}$ and magnetic $\boldsymbol{B}$ fields (magnitudes and directions) in the rest frame K of the wire. Use cylindrical coordinates, taking the z axis along the wire, denoting the radial distance from the wire by s, and the azimuthal angle by ϕ.

(b) [**2 points**] What are the electric V and magnetic $\boldsymbol{A}$ potentials in the frame K?
You can set the electric potential to vanish at $s = s_0$, where s_0 is arbitrary.

Another reference frame K' moves relative to the frame K with a velocity $\boldsymbol{v}$ *parallel* to the direction of the wire. The goal of the remaining problem is to compute the electric and magnetic fields in the new frame K' using *three* different approaches and compare the results. Express your answers in terms of $\beta = v/c$ and $\gamma = 1/\sqrt{1-\beta^2}$, where c is the speed of light.

(c) [**4 points**] Using the Lorentz transformation of the potentials (and the coordinates), find the potentials V' and $\boldsymbol{A'}$ in the frame K' from the potentials obtained in the original frame K in Part (b).

(d) [**4 points**] Using the potentials from Part (c), calculate the fields $\boldsymbol{E'}$ and $\boldsymbol{B'}$ in the frame K'.

(e) [**4 points**] Next, using the Lorentz transformation properties of the *fields*, obtain $\boldsymbol{E'}$ and $\boldsymbol{B'}$ in the frame K' directly from the fields computed in the frame K in Part (a).

Compare with the result of Part (d).

(f) [**3 points**] What are the linear charge density λ' and the line current I' associated with the wire in the frame K'?

(g) [**4 points**] From the charge and current densities in the frame K' obtained in Part (f), directly calculate the fields $\boldsymbol{E'}$ and $\boldsymbol{B'}$ in this frame.

Compare with the results of Parts (d) and (e).

Possibly useful formulae

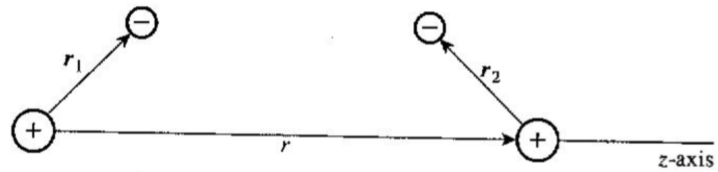
The Lorentz transformation of the fields: $\boldsymbol{E}'_\perp = \gamma\Big[\boldsymbol{E}_\perp + (\boldsymbol{v}\times\boldsymbol{B})\Big]$; $\boldsymbol{B}'_\perp = \gamma\Big[\boldsymbol{B}_\perp - \frac{1}{c^2}\,(\boldsymbol{v}\times\boldsymbol{E})\Big]$, where $\perp$ denotes the components perpendicular to $\boldsymbol{v}$.

Curl of a vector field $\boldsymbol{F}$ in cylindrical coordinates:

$$\boldsymbol{\nabla}\times\boldsymbol{F} = \left[\frac{1}{s}\frac{\partial F_z}{\partial\phi} - \frac{\partial F_\phi}{\partial z}\right]\hat{\boldsymbol{s}} + \left[\frac{\partial F_s}{\partial z} - \frac{\partial F_z}{\partial s}\right]\hat{\boldsymbol{\phi}} + \frac{1}{s}\left[\frac{\partial\,(sF_\phi)}{\partial s} - \frac{\partial F_s}{\partial\phi}\right]\hat{\boldsymbol{z}}$$

Problem B.3

This problem studies the van der Waals interaction between two electrically neutral atoms. Each atom consists of one electron and one proton, as shown in the Figure. The vector $\boldsymbol{r}$ connects one proton to another. The distance r is much bigger than the atoms' sizes.



Assume that protons' positions are fixed, and each electron is bound to the corresponding proton by a 3D isotropic harmonic-oscillator potential (instead of the Coulomb potential)

$$H_0 = \frac{\boldsymbol{p}_1^2}{2m} + \frac{m\omega^2 \boldsymbol{r}_1{}^2}{2} + \frac{\boldsymbol{p}_2^2}{2m} + \frac{m\omega^2 \boldsymbol{r}_2{}^2}{2}. \tag{1}$$

Here $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ are the coordinates of the electrons relative to their protons, $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$ are the corresponding momenta, m is the electron mass, and ω is the oscillator frequency. For simplicity, treat the electrons as distinguishable and ignore their spins.

(a) **[2 points]** The electric dipole moments of the atoms, $\boldsymbol{d}_1 = e\boldsymbol{r}_1$ and $\boldsymbol{d}_2 = e\boldsymbol{r}_2$, experience the dipole-dipole interaction, where $\hat{\boldsymbol{r}} = \boldsymbol{r}/r$ is the unit vector along $\boldsymbol{r}$,

$$V = \frac{\boldsymbol{d}_1 \cdot \boldsymbol{d}_2 - 3(\boldsymbol{d}_1 \cdot \hat{\boldsymbol{r}})(\boldsymbol{d}_2 \cdot \hat{\boldsymbol{r}})}{4\pi\epsilon_0\, r^3}. \tag{2}$$

Taking the z axis along $\hat{\boldsymbol{r}}$, and the x and y axes perpendicular, express Eq. (2) in terms of the Cartesian coordinates $\boldsymbol{r}_1 = (x_1, y_1, z_1)$ and $\boldsymbol{r}_2 = (x_2, y_2, z_2)$.

(b) **[9 points]** Representing a 3D oscillator in Eq. (1) as a sum of three 1D oscillators, the energy eigenstates of H_0 can be labeled by the quantum numbers n of the oscillators

$$|k\rangle = |n_x^{(1)}, n_y^{(1)}, n_z^{(1)}\rangle |n_x^{(2)}, n_y^{(2)}, n_z^{(2)}\rangle. \tag{3}$$

Calculate matrix elements of V between the excited states $|k\rangle$ and the ground state $|0\rangle$ of the oscillators and identify the states $|k\rangle$ giving nonzero results.

(c) **[10 points]** Treating V as a perturbation to H_0, find the lowest-order correction $U(r)$ to the ground state energy. How does $U(r)$ depend on r? Is $U(r)$ attractive or repulsive?

(d) **[2 points]** Describe qualitatively without detailed calculations how the conclusions of Part (c) about the dependence of $U(r)$ on r and its attractiveness or repulsiveness would be affected, if a more realistic Coulomb potential is used for H_0 in Eq. (1).

(e) **[2 points]** Now suppose the two atoms have a relative orbital angular momentum l. Write the energy $U_l(r)$ of this orbital motion in terms of the mass M of each atom and the angular quantum number l.

Is the combined interaction energy $U(r) + U_l(r)$ between the two atoms attractive or repulsive at a long distance r for $l = 0$ and $l \neq 0$?

Useful info: For a 1D harmonic oscillator, the matrix element is $\langle 1|x|0\rangle = \sqrt{\hbar/2m\omega}$.

Problem B.4

An electron is in the excited state $|2,1,1\rangle$ of the hydrogen atom, in the notation $|n,l,m\rangle$ for a given angular momentum quantization axis z. By spontaneously emitting a photon, the electron can make a transition to the ground state $|1,0,0\rangle$. This problem studies the lifetime τ of the initial state.

The vector potential $\boldsymbol{A}$ for the electromagnetic field is represented by the operator

$$\boldsymbol{A} = \sum_{\boldsymbol{k}} \sum_{\lambda=1,2} \sqrt{\frac{2\pi\hbar c}{kV}} \left(a_{\boldsymbol{k},\lambda} \boldsymbol{\varepsilon}_{\boldsymbol{k},\lambda} e^{i\boldsymbol{k}\cdot\boldsymbol{r}} + a^\dagger_{\boldsymbol{k},\lambda} \boldsymbol{\varepsilon}^*_{\boldsymbol{k},\lambda} e^{-i\boldsymbol{k}\cdot\boldsymbol{r}} \right), \tag{1}$$

where V is the volume of a large "box" containing the system, and the operators $a^\dagger_{\boldsymbol{k},\lambda}$ and $a_{\boldsymbol{k},\lambda}$ create and destroy photons with wave vectors $\boldsymbol{k}$ and polarizations $\boldsymbol{\varepsilon}_{\boldsymbol{k},\lambda}$ labeled by $\lambda = 1, 2$.

The Hamiltonian of electron-photon interaction is $H_{\text{int}} = -(e/m_e c)\, \boldsymbol{p} \cdot \boldsymbol{A}$, where c is the speed of light, and e, m_e and $\boldsymbol{p}$ are the electron charge, mass, and momentum operator.

(a) [**5 points**] Calculate the matrix elements $\boldsymbol{M} = \langle 2,1,1|\boldsymbol{p}|1,0,0\rangle$ of the electron momentum operator between the excited and the ground states in terms of the Bohr radius a using the wave functions given below. Obtain all three components of $\boldsymbol{M} = (M_x, M_y, M_z)$ and indicate which of them vanish by symmetry.

Even if you do not manage to calculate the values of nonzero matrix elements, you can still proceed to the next parts using them as given.

(b) [**2 points**] Formulate the electric dipole approximation to Eq. (1) for the electron-photon interaction H_{int}. Specify when it is applicable to this problem and use it below.

(c) [**5 points**] Using the Fermi golden rule, obtain the rates of spontaneous emission of a photon with the wave vector $\boldsymbol{k}$ and polarization $\boldsymbol{\varepsilon}_{\boldsymbol{k},\lambda}$ for two linear polarizations: $\boldsymbol{\varepsilon}_{\boldsymbol{k},1}$ in the plane of $\boldsymbol{k}$ and z, and $\boldsymbol{\varepsilon}_{\boldsymbol{k},2}$ perpendicular to the plane of $\boldsymbol{k}$ and z.

Express your answers in terms of the matrix elements $M_{x,y,z}$ and the Dirac delta-function for energy conservation. What is the frequency ω of the emitted photon?

(d) [**5 points**] Next, integrate the result of Part (c) over the absolute value k, thus eliminating the energy delta-function, and obtain the transition rates $d\Gamma_1/d\Omega$ and $d\Gamma_2/d\Omega$ for a small solid angle $d\Omega$ around the vector $\boldsymbol{k}$ for each polarization.

Describe how $d\Gamma_1/d\Omega$ and $d\Gamma_2/d\Omega$ depend on the angle θ between the z axis and $\boldsymbol{k}$.

(e) [**5 points**] Finally, integrate the result of Part (d) over $d\Omega$ and obtain the total rate of spontaneous emission $\Gamma = \Gamma_1 + \Gamma_2$ and the lifetime $\tau = 1/\Gamma$ of the excited state.

(f) [**3 points**] Now consider the lifetime of an excited state $|2,1,m\rangle$ with a different value of m. Would it be different from the τ calculated above? Explain without calculations.

The wave functions and energies of the hydrogen atom, where a is the Bohr radius:

$$\psi_{1,0,0} = \frac{1}{\sqrt{\pi a^3}}\, e^{-r/a}, \qquad \psi_{2,1,1} = \frac{1}{8\sqrt{\pi a^3}}\, \frac{x+iy}{a}\, e^{-r/2a}, \qquad E_n = -\frac{e^2}{2n^2 a}, \qquad a = \frac{\hbar^2}{m_e e^2}. \tag{2}$$

Useful integral: $\int_0^\infty u^n\, e^{-u/b}\, du = n!\, b^{n+1}$.