UNIVERSITY OF MARYLAND
Department of Physics
College Park, Maryland

PHYSICS Ph.D. QUALIFYING EXAMINATION
PART I

August 22, 2018 9:00 a.m. – 1:00 p.m.

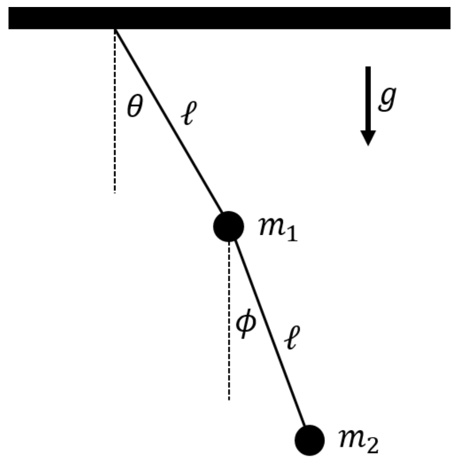
Do any four problems. Each problem is worth 25 points.
Start each problem on a new sheet of paper (because different faculty members will be grading each problem in parallel).

Be sure to write your Qualifier ID ("control number") at the top of each sheet — not your name! — and turn in solutions to four problems only. (If five solutions are turned in, we will only grade # 1 - # 4.)

At the end of the exam, when you are turning in your papers, please fill in a "no answer" placeholder form for the problem that you skipped, so that the grader for that problem will have something from every student.

You may keep this packet with the questions after the exam.

Problem I.1



Consider a double pendulum made with masses m_1 and m_2 hanging from a fixed top support by identical massless rods of length ℓ, as shown. A uniform gravitational acceleration g acts on this system.

(a) [**4 points**] Determine the Lagrangian of the system in terms of the angle coordinates θ and ϕ. (Do not assume that the angles are small. You may use appropriate Cartesian coordinates at first, but then convert to the angle coordinates.) Use $m_T \equiv (m_1 + m_2)$ for convenience.

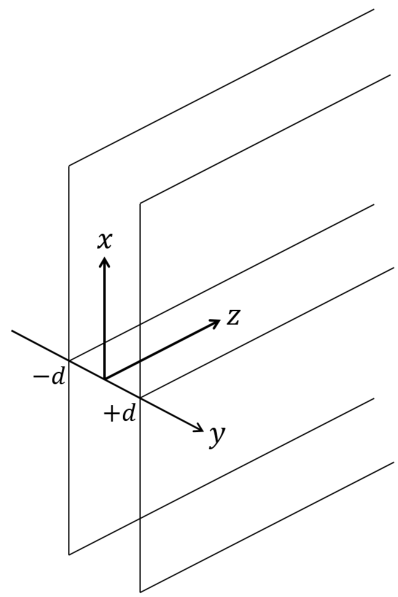
(b) [**6 points**] Now, making the approximation that both angles are small, use the Lagrangian to determine coupled equations of motion for the angle coordinates in reasonably simple form.

(c) [**3 points**] Qualitatively describe the possible normal-mode (periodic) motions of the system.

(d) [**8 points**] Determine the frequencies of the normal modes, still assuming small-angle oscillations.

(e) [**4 points**] Qualitatively, interpreting your part (d) answer, what happens to the normal-mode frequencies if you decrease $m_1 \to 0$ while keeping m_2 fixed?

Problem I.2



The figure shows a slab of dielectric material which extends to $\pm\infty$ in the x and z directions but extends only from $-d$ to $+d$ in the y direction. In SI units, the slab has dielectric constant $\epsilon > \epsilon_0$ and is nonmagnetic. Air ($\epsilon \approx \epsilon_0$) surrounds the slab. We consider an electromagnetic wave propagating in the z direction through the slab. The Maxwell Equations governing this system are given in SI units as

$$\partial_t \mathbf{B} = -\nabla \times \mathbf{E}, \quad \partial_\mathbf{t} \epsilon \mathbf{E} = \nabla \times \mathbf{B}/\mu_\mathbf{0}, \quad \textbf{where} \;\; \epsilon_\mathbf{0}\mu_\mathbf{0} = \mathbf{1/c^2}. \tag{1}$$

Assume, for this mode, that $B_y = B_z = 0$, and that the other field components (B_x and the electric field vector $\mathbf{E}$) *do not vary with* x. In this problem you are going to show that, with certain constraints, the wave is *guided* in the z direction, in the sense that the fields decrease rapidly with $|y|$ outside the dielectric slab.

(a) [**5 points**] Starting from the Maxwell equations, write down, *in Cartesian coordinates*, the equation for the time evolution of B_x. Assume constant ϵ. Noting that the B_x equation couples to two particular components of $\mathbf{E}$, write down the equations for the time evolution of these coupled $\mathbf{E}$ components, and observe that the resulting system of equations is closed. Combine these equations to obtain a partial differential equation for B_x alone.

(b) [**4 points**] Assume that the wave propagates in the waveguide at a given frequency ω and with a given wavenumber k in the z-direction. Let $B_x \to B_x(y)e^{ikz-i\omega t}$ and thus deduce the ordinary differential equation that must be satisfied by $B_x(y)$, separately inside and outside the slab. Deduce also how each coupled component of $\mathbf{E}(\mathbf{y})$ is related to $B_x(y)$, for given ω and k.

(c) [**4 points**] From the general Maxwell equations in dielectric media given above, state the boundary ("pillbox") conditions satisfied by $\mathbf{E}$ across the slab boundary $y = d$.

Apply these for each of the coupled **E** variables involved in our wave and thus deduce the boundary conditions on B_x and its derivative across the discontinuity.

(d) [**4 points**] Up to a constant, write down a solution for $B_x(y)$ outside the slab. For a guided wave, the solution must decay exponentially to zero as y becomes large. What condition does this place on k and ω?

(e) [**4 points**] Assuming even solutions about $y = 0$, write down, up to a constant, a solution for $B_x(y)$ inside the slab. What condition does this place on k, ω, and ϵ?

(f) [**4 points**] Apply the boundary conditions at $z = d$ obtained in (c) and, so, find a dispersion relation of the form $(\epsilon/\epsilon_0)\beta = \alpha \tan(\alpha d)$, where α and β are functions of ω and k. Specify both these functions.

Problem I.3

A small particle bound to a surface defect by a centrally attractive force may act as a two-dimensional classical harmonic oscillator. Assume that this particle is in thermal equilibrium with its environment, which has temperature T.
(Note that the parts of this problem are not all sequential; if you get stuck on one part, you may still be able to do some of the later parts.)

(a) [**3 points**] Suppose the restoring force on the particle produces a natural (angular) frequency ω for linear oscillations in either the x direction or the y direction. Write down (you don't need to derive it) the Hamiltonian of the particle as a function of its mass m, instantaneous position (x, y), and momentum $\mathbf{p}$.

(b) [**6 points**] Calculate the partition function for this system, doing integrals where appropriate to reduce it to a simple form involving T. (Recall that the partition function involves Planck's constant even for classical systems, as a conventional unit of phase space.)

(c) [**5 points**] Now consider an ensemble of N of these two-dimensional harmonic oscillators, taking them to be non-interacting and distinguishable. Find the Helmholtz free energy and the entropy of the ensemble in terms of T, ω, and constants.

(d) [**2 points**] Now go back to focusing on just a *single* bound classical particle. While it is in thermal equilibrium with its environment, its energy will *not* be constant over time. Explain why in one or two sentences.

(e) [**4 points**] Use the partition function from part (b) to calculate the average energy of this (single) bound-particle system expressed in terms of T. (Or, for partial credit, determine its average energy in some other way.)

(f) [**5 points**] Calculate the root-mean-square fluctuation in the energy of this system, expressed in terms of T.

Possibly useful:

$$\int_{-\infty}^{\infty} e^{-ax^2}\,dx = \sqrt{\frac{\pi}{a}} \tag{1}$$

$$\int_{0}^{\infty} x e^{-ax^2}\,dx = \frac{1}{2a} \tag{2}$$

$$\int_{-\infty}^{\infty} x^2 e^{-ax^2}\,dx = \frac{1}{2a}\sqrt{\frac{\pi}{a}} \tag{3}$$

$$\int_{0}^{\infty} x^3 e^{-ax^2}\,dx = \frac{1}{2a^2} \tag{4}$$

$$\int_{-\infty}^{\infty} x^4 e^{-ax^2}\,dx = \frac{3}{4a^2}\sqrt{\frac{\pi}{a}} \tag{5}$$

Problem I.4

The $W^{\pm}$ bosons were first discovered in the collisions of beams of protons and antiprotons. The two beams circulate in opposite directions in a large ring and have the same energy, E_{b}, which is much greater than the proton rest energy. The reaction is understood to be a collision between a **u** quark from the proton and an anti-**d** quark from the antiproton. The quarks can carry any fraction of the beam energy, have essentially zero mass, and essentially zero momentum transverse to the beam axis.

(a) **[5 points]** Let x_1 be the fraction of the proton's momentum carried by the **u** quark, and let x_2 be the fraction of the antiproton's momentum carried by the anti-**d** quark. (Both x_1 and x_2 are between 0 and 1.) Determine the necessary relationship between x_1 and x_2 such that they annihilate, producing a W particle (with mass M_{W}) and nothing else. This relationship should be in terms of M_{W} and the beam energy.

(b) **[5 points]** Based on your answer to (a), what is the permissible range of values for the W particle's momentum component parallel to the proton beam axis? That is, calculate p_{max}.

(c) **[5 points]** A variable commonly used to characterize a particle emerging from a beam-beam collision is the "rapidity",

$$\eta \equiv \frac{1}{2} \ln \left(\frac{E + p_L}{E - p_L} \right)$$

where p_L is the component of the particle's momentum parallel to the proton beam axis and E is the energy of the particle. What is the maximum value of η possible for a W particle produced in the collision described above (in the lab frame)?

(In this scenario, its momentum is much larger than its mass, but M_{W} cannot be totally neglected. Expand to lowest order and simplify to get a finite (approximate) value for the maximum η in terms of M_{W} and p_{max}.)

(d) **[5 points]** The collision will actually produce multiple particles, and the W will decay almost instantly to other particles. Rapidity is a frame-dependent quantity (which is why we specified the lab frame in the previous part). However, when two particles emerge from the same collision, the rapidity *difference* $\Delta\eta \equiv \eta_1 - \eta_2$ is invariant under Lorentz transformations along the beam axis. Prove that explicitly.

(e) **[5 points]** Many collider detectors installed at proton-antiproton collision points have a solenoidal magnet centered on the interaction point and co-axial with the beam axis, and either a silicon or wire-chamber tracking detector to record the paths of charged particles emerging from the collision. Explain how the path of a charged particle in this region is used to determine the p_L and E quantities used to calculate its rapidity (ignoring, in this case, any information that may be available from a calorimeter). (Hint: consider the direction of the magnetic field produced by a solenoid.)

Problem I.5

A uniform string of length L under tension τ undergoes small transverse oscillations. The mass per unit length of the string is given by μ, and the equilibrium position of the string lies along the x axis. The transverse displacement of the string at the point with coordinate x at time t is denoted by $y(x,t)$. One end of the string at $x = 0$ is attached to a fixed support so that the transverse displacement at this point vanishes, $y(0,t) = 0$. The other end of the string is attached to a point particle of mass m that is restricted to lie along the line $x = L$, but is free to move without friction along the y direction.

(a) [**4 points**] Write down the wave equation of motion for small amplitude displacements $y(x,t)$. Express the velocity of propagation of transverse waves in terms of τ and μ.

(b) [**5 points**] By applying Newton's 2^{nd} Law to the mass m, show that the appropriate boundary condition for *small* displacements along y at $x = L$ has the form

$$\kappa \frac{\partial y}{\partial x} = -\frac{\partial^2 y}{\partial t^2} \,. \tag{1}$$

Express the constant κ in terms of the physical parameters in the problem.

(c) [**10 points**] Use the boundary condition above to obtain a transcendental equation that implicitly determines the characteristic frequencies of the normal modes of this system. (You may write the equation in terms of a wavenumber k instead of a frequency parameter).

Note: If you can't get the answer to this part, you can still answer parts (d) and (e) through other lines of reasoning for partial credit.

(d) [**3 points**] Use this transcendental equation to obtain the solution for the wavelengths of the normal modes in the limit that $m \to \infty$, (or, more precisely, $m \gg \mu L$). Give a physical interpretation of your result.

(e) [**3 points**] Use the equation from part (c) to obtain the solution for the wavelengths of the normal modes in the limit that $m \to 0$. Give a physical interpretation of your result.