UNIVERSITY OF MARYLAND
Department of Physics
College Park, Maryland

PHYSICS Ph.D. QUALIFYING EXAMINATION
PART I

9:00 a.m. – 1:00 p.m.

Do any four problems. Each problem is worth 25 points.
Start each problem on a new sheet of paper (because different faculty members will be grading each problem in parallel).

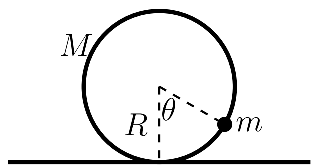
Be sure to write your Qualifier ID ("control number") at the top of each sheet — not your name! — and turn in solutions to four problems only. (If five solutions are turned in, we will only grade # 1 - # 4.)

At the end of the exam, when you are turning in your papers, please fill in a "no answer" placeholder form for the problem that you skipped, so that the grader for that problem will have something from every student.

You may keep this packet with the questions after the exam.

Problem I.1

A cylindrical hoop of negligible thickness, mass M, and radius R has a point mass m attached to its circumference. The hoop rolls without sliding along a horizontal plane under the influence of gravity (the gravitational acceleration is g).



(a) **[5 points]** Show that the kinetic energy of the system is

$$K = MR^2\dot{\theta}^2 + mR^2\dot{\theta}^2(1 - \cos\theta), \tag{1}$$

where θ is the angle between the vertical direction and the line connecting the center of the hoop to the mass m.

(b) **[5 points]** Obtain the Lagrangian of the system and the equation of motion for the variable θ.

(c) **[5 points]** Keeping terms up to order θ and $\dot{\theta}$ in the equation of motion, determine the angular frequency ω of small oscillations around the equilibrium position. What is the behavior of ω in the small M limit?

(d) **[5 points]** Return to part (b) and explicitly set $M = 0$. What is the equation of motion for small oscillations in this case? Find an expression for the dependence of ω on the amplitude θ_0.

(e) **[5 points]** The angular frequency obtained in the limit $M \to 0$ in parts (c) and (d) are different. Examine the implicit assumptions made in each case to reconcile this apparent discrepancy.

Problem I.2

(a) **[6 points]** A charge $+q$ is located at $(x, y, z) = (0, 0, h)$ above an infinite grounded conducting plate located in the $x-y$ plane at $z = 0$. Calculate the electric potential for $z > 0$ by the method of images. Show clearly that your solution satisfies the boundary condition of zero tangential $\boldsymbol{E}$ field at the plate. What is the force on the charge $+q$ due to induced charges on the conducting plate? [Use $k = 1/(4\pi\epsilon_0)$.]

(b) **[6 points]** Suppose we replace the $+q$ charge with a permanent point dipole $\boldsymbol{p} = p_0\hat{\boldsymbol{z}}$. By modeling the dipole as two opposite closely spaced charges, deduce which way the image dipole points. Using the formulae provided below, calculate the electric potential of the system for $z > 0$. Show clearly that your solution satisfies the boundary condition of zero tangential $\boldsymbol{E}$ field at the plate, and that it is consistent with your deduced dipole directions.

(c) **[6 points]** Calculate the $\boldsymbol{E}$ field at $z = h$ due to the image dipole $\boldsymbol{p}_0$. Which way is the force between the dipoles?

(d) **[7 points]** Suppose we replace the $+q$ charge from part (a) with a spherical, neutral conductor of radius a. In what follows, assume $a << h$. You are reminded that, in the presence of any applied electric field, the spherical conductor will develop a dipole moment $\boldsymbol{p} = \boldsymbol{E}_{\text{applied}} a^3/k$. Suppose such a dipole moment is formed, in the $\hat{z}$ direction, as an initial fluctuation. From your results (b) and (c), calculate the ratio of the E field (at $z = h$) from the induced dipole to the fluctuation E_{applied} field?.

Potentially useful: The electric potential due to a point dipole $\boldsymbol{p}$, at position vector $\boldsymbol{r}$ from the dipole center, is given by $\phi(\boldsymbol{r}) = k\boldsymbol{p}\cdot\boldsymbol{r}/r^3$. The electric field resulting from this potential is $\boldsymbol{E}(\boldsymbol{r}) = (k/r^3)[3\hat{\boldsymbol{r}}(\boldsymbol{p}\cdot\hat{r}) - \boldsymbol{p}]$.

Problem I.3

A small particle bound to a surface defect by a centrally attractive force may act as a two-dimensional classical harmonic oscillator. Suppose the restoring force on the particle produces a natural (angular) frequency ω for linear oscillations in either the x direction or the y direction. Assume that a collection of N particles of this sort acting classically are in thermal equilibrium with their environment, which has temperature T.

The partition function for such a system is given by $Z_N = \left(\frac{k_B T}{\hbar\omega}\right)^{2N}$. Note that although the system is classical, the expression contains an $\hbar$. This is merely a conventional factor that arises in the normalization of phase space and picked so that the quantum expression maps onto the classical one.

(a) [**5 points**] Consider on a *single* bound classical particle. While it is in thermal equilibrium with its environment, its energy will *not* be constant over time—the particle can exchange energy with the heat bath. However its average energy is determined. Calculate the average energy of this (single) bound-particle system expressed in terms of T.

(b) [**5 points**] Calculate the root-mean-square fluctuation in the energy of this single particle system, expressed in terms of T.

(c) [**6 points**] Find the Helmholtz free energy and the entropy of the ensemble N particles in terms of T, ω, and constants.

In the remainder of this problem you will derive $Z_N = \left(\frac{k_B T}{\hbar\omega}\right)^{2N}$.

(d) [**7 points**] Calculate the partition function for a single particle system in equilibrium with the heat bath , doing integrals where appropriate to reduce it to a simple form involving T. (Recall that the partition function involves Planck's constant even for classical systems, as a conventional unit of phase space.)

(e) [**2 points**] From Part d, derive the partition function for N particles.

Possibly useful:

$$\int_{-\infty}^{\infty} e^{-ax^2}dx = \sqrt{\frac{\pi}{a}} \qquad \int_{0}^{\infty} xe^{-ax^2}dx = \frac{1}{2a} \qquad \int_{-\infty}^{\infty} x^2e^{-ax^2}dx = \frac{1}{2a}\sqrt{\frac{\pi}{a}}$$

$$\int_{0}^{\infty} x^3e^{-ax^2}dx = \frac{1}{2a^2} \qquad \int_{-\infty}^{\infty} x^4e^{-ax^2}dx = \frac{3}{4a^2}\sqrt{\frac{\pi}{a}}$$

Problem I.4

The 'Breakthrough Starshot' program hopes to accomplish space travel without on-board fuel by shining a laser beam from Earth on the spacecraft. We analyze this scenario for two separate cases.

(a) **[9 points]** Let us first consider the increase in mass of the ship when the photons are completely absorbed by it. We begin with the more general case of a moving particle of rest mass μ colliding with a stationary object of rest mass m_0 and sticking to it. Derive an expression for the resultant mass-squared m^2 as a function of μ, m_0, and ϵ, where $\epsilon = \mu c^2 + K$ is the relativistic energy and K is the kinetic energy of the incoming particle. How does the increase in the rest mass-squared, $m^2 - (m_\mu + m_0)^2$, depend on K?

(b) **[7 points]** Now consider the case when the incoming particle is a photon and let m_0 be the rest mass of the spaceship. Consider the frame in which the spaceship is at rest before absorbing the photon, and has velocity v after absorbing the photon. The mass of the spaceship after it has absorbed the photon becomes $m = \alpha(\beta) m_0$, where $\beta = v/c$. Derive the expression for $\alpha(\beta)$, which involves only the velocity v.

(c) **[2 points]** Explain why adding a mirror to the spaceship would improve the efficiency of this system for accelerating the spacecraft, for a given energy of the incoming photons.

(d) **[7 points]** When the spacecraft is already moving, it can be shown that the 4-momentum transferred by a reflected photon is

$$\boldsymbol{t} = 2\left(\boldsymbol{p}_p - (\boldsymbol{p}_p \cdot \boldsymbol{p}_s)\frac{\boldsymbol{p}_s}{(mc)^2}\right) \tag{1}$$

where $\boldsymbol{p}_p$ and $\boldsymbol{p}_s$ are the 4-momenta (before the reflection) of the photon and the spaceship, respectively. Find the energy transferred by evaluating a suitable component of $\boldsymbol{t}$ in the Earth frame. Then evaluate the "efficiency per photon" of this propulsion system, defined as the ratio

$$\frac{\text{energy gain by spacecraft from reflection of photon}}{\text{energy of photon at incidence}} \tag{2}$$

all energies being measured in the Earth frame. In the limit $\beta \to 1$, this efficiency approaches unity. What is the physical explanation for this?

Possibly useful information: A common form of a momentum 4-vector is $\boldsymbol{p} = (E/c, \vec{p})$ where E and $\vec{p}$ are the total energy and 3-vector momentum of the particle.

Problem I.5

Two parallel semi-transparent mirrors, separated a distance d, enclose a non-dispersive material of refraction index n. Consider electromagnetic waves of radial frequency ω propagating at normal incidence to the mirror plane.

(a) **[2 points]** What is the optical phase θ accumulated during propagation from one mirror to the other, in terms of d, n, ω, and the speed of light c?

(b) **[2 points]** Each mirror has an electric-field transmission amplitude t and a reflection amplitude r. Assuming the mirrors are lossless, use conservation of energy to relate these two quantities.

(c) **[5 points]** A continuous beam of coherent light which passes through the first mirror will bounce back and forth between the mirrors, interfering with new incoming light each time it bounces on the first mirror. This arrangement is called a Fabry-Perot cavity. Considering the effects of interference, show that the power transmission coefficient (ratio of transmitted to incident electromagnetic intensity) is $T = t^4/(1 + r^4 - 2r^2 \cos 2\theta)$.

(d) **[3 points]** Assume that the mirrors are highly reflective so that $r \to 1$. What is the transmission coefficient when $\omega = \omega_0 = \frac{\pi c}{nd}$?

(e) **[3 points]** Use an appropriate approximation to find the functional dependence of $T(\omega)$ in the vicinity of ω_0. You may express it in terms of $\delta\omega \equiv \omega - \omega_0$.

(f) **[4 points]** Again assuming nearly perfect reflection, determine the "quality factor" for this resonance, Q, in terms of r.