UNIVERSITY OF MARYLAND
Department of Physics
College Park, Maryland

PHYSICS Ph.D. QUALIFYING EXAMINATION
PART II

9:00 a.m. – 1:00 p.m.

Do any four problems. Each problem is worth 25 points.
Start each problem on a new sheet of paper (because different faculty members will be grading each problem in parallel).

Be sure to write your Qualifier ID ("control number") at the top of each sheet — not your name! — and turn in solutions to four problems only. (If five solutions are turned in, we will only grade # 1 - # 4.)

At the end of the exam, when you are turning in your papers, please fill in a "no answer" placeholder form for the problem that you skipped, so that the grader for that problem will have something from every student.

You may keep this packet with the questions after the exam.

Problem II.1

The Schrödinger equation for the helium atom cannot be solved exactly. However, if we replace each of the Coulomb forces by a spring force, the system *can* be solved exactly. As an example, consider the Hamiltonian H in 3-dimensional space given by

$$H = -\frac{\hbar^2}{2m}\left(\nabla_1^2 + \nabla_2^2\right) + \frac{1}{2}m\omega^2\left(r_1^2 + r_2^2\right) - \frac{\lambda}{4}m\omega^2|\mathbf{r}_1 - \mathbf{r}_2|^2 \ , \tag{1}$$

with $0 < \lambda < 1$. Here $\mathbf{r}_1$ and $\mathbf{r}_2$ represent the coordinates associated with the two electrons and m is the electron mass.

(a) [**5 points**] Neglecting the last term (i.e. setting $\lambda = 0$) in H, determine the ground state energy of the two (no longer coupled) 3D harmonic oscillators. Also write down the ground state wave function.

(b) [**5 points**] Use this uncoupled ground state wave function and first order perturbation theory to estimate the ground state energy of the full, coupled Hamiltonian H.

(c) [**10 points**] By a suitable change of variables, show how the full Hamiltonian H can be transformed into the sum of two independent simple harmonic oscillators in 3D,

$$H = \left[-\frac{\hbar^2}{2m}\nabla_u^2 + \frac{1}{2}m\omega^2 u^2\right] + \left[-\frac{\hbar^2}{2m}\nabla_v^2 + \frac{1}{2}\left(1-\lambda\right)m\omega^2 v^2\right] \ . \tag{2}$$

Explicitly give the $u = u(\mathbf{r}_1, \mathbf{r}_2)$ and $v = v(\mathbf{r}_1, \mathbf{r}_2)$ that satisfy this transformation.

(d) [**5 points**] Determine the exact ground state energy of the system. How well does the answer agree with your estimate from part (b) above?

Note: The normalized ground state wave function of a single harmonic oscillator in one dimension is given by

$$\psi(x) = \left(\frac{\alpha}{\pi}\right)^{\frac{1}{4}} \exp\left(-\alpha x^2/2\right) \ , \tag{3}$$

where $\alpha = m\omega/\hbar$.

Problem II.2

An electron of mass m moves in a one-dimensional attractive potential $U(x) = -\lambda\delta(x)$, where $\delta(x)$ is the Dirac delta function and $\lambda > 0$.

(a) [**5 points**] Find the wave function and the energy E_0 of the bound state. What is the parity of the wave function with respect to the operation $x \to -x$?

(b) [**5 points**] Find the wave functions and the energies of the unbound states which are antisymmetric with respect to the parity operation $x \to -x$. Because they are not square-integrable, normalize them such that total $|\psi|^2$ in one wavelength is unity.

For time $t < 0$, the electron is in the ground state of the potential. At time $t = 0$, a small AC electric field $\mathcal{E}(t) = \mathcal{E}_0 \sin(\omega t)$ with frequency $\omega > |E_0|/\hbar$ is turned on. The Hamiltonian of the perturbation is

$$V = -2ex\mathcal{E}_0 \sin(\omega t)$$

where e is the electron charge. The perturbation may cause a transition from the bound state to one of the unbound states.

(c) [**5 points**] Calculate the nonvanishing matrix elements of the perturbation between the ground state and the unbound states.

(d) [**5 points**] Using the Fermi golden rule, calculate the transition rate. Make sure the dimensionality of your final result is 1/time.

(e) [**5 points**] Sketch how the ionization rate depends on the frequency ω.

Potentially useful: $\int_0^\infty dx\, x \sin(ax) e^{-bx} = \frac{2ab}{(a^2+b^2)^2}$

Problem II.3

A spin-1/2 particle of mass m moves in the x direction in a potential given by:

$$V(x) = V_0\sigma_z \text{ for } x \geq 0 \qquad \text{and} \qquad V(x) = 0 \text{ for } x < 0. \tag{1}$$

Here σ_z is one of the Pauli matrices

$$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \qquad \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

acting on the spinor wavefunction of the particle, where the quantization axis is along z.

(a) [**2 points**] Describe qualitatively how the potential (1) can be realized experimentally.

(b) [**3 points**] Consider a beam of particles coming from $x = -\infty$ with energy $E > V_0 > 0$ and velocity in the positive x direction. Suppose the particles are in an eigenstate of σ_x with eigenvalue $+1$. Write down the general wavefunction $\psi(x,t)$ of the incoming beam as a two-component spinor, where the quantization axis is along z.

(c) [**3 points**] Write down general expressions for the spinor wavefunctions of the transmitted and reflected beams with yet unknown amplitudes to be determined later.

(d) [**4 points**] Using the boundary conditions at $x = 0$, obtain equations connecting the spinor amplitudes of the incoming, reflected, and transmitted beams.

(e) [**3 points**] Solve the equations obtained above and express the spinor amplitudes of the transmitted and reflected beams in terms of the incoming beam amplitude.

(f) [**4 points**] Using the above result, calculate the transmission and reflection coefficients T and R. They are defined as the transmitted and reflected probability fluxes divided by the incoming probability flux. Keep in mind that the probability flux depends on velocity, which is different for $x < 0$ and $x > 0$. Check that $T + R = 1$.

(g) [**3 points**] Determine spin polarization, i.e., the direction of spin, of the reflected beam. It is characterized by the polar angle θ_r relative to the axis z and the azimuthal angle φ_r in the (x,y) plane. Is the reflected beam still polarized along the axis x?

(h) [**3 points**] Determine spin polarization of the transmitted beam as a function of x, characterized by the polar angle $\theta_t(x)$ and azimuthal angle $\varphi_t(x)$. Describe spatial variation pattern of spin polarization and determine the spatial period L.

Problem II.4

The wave-function for a spin-1/2 particle is written as a two-component spinor

$$\Psi(x) = \begin{pmatrix} \psi_\uparrow(x) \\ \psi_\downarrow(x) \end{pmatrix}.$$

The time-reversal operator for this system is written as $\Theta = i\sigma_y K$ where K is the complex conjugation operator. Consider the single-particle Hamiltonian

$$H = ap^4 + bp^3\sigma_x + cp\sigma_z - de^{-x^2},$$

where x is the position operator, p is the momentum operator, and $a, b, c, d \geq 0$ are constants.

(a) [**8 points**] By considering the action of the operators p, σ_x, σ_z and e^{-x^2} on a wave-function $\Psi(x)$ detemine whether each of these operators are even or odd (i.e. symmetric or anti-symmetric) under time-reversal.

(b) [**2 points**] Use the symmetries of the operators in the last part to show that H is symmetric (even) under time-reversal.

(c) [**5 points**] By considering the action of Θ on the wave-function $\Psi(x)$ show that $\Theta^2 = -1$.

(d) [**5 points**] (Kramer's theorem) Consider an energy eigenstate $\Psi_n(x)$ with energy eigenvalue E_n for the Hamiltonian of this problem. Show that $\Psi'_n(x) = \Theta\Psi_n(x)$ is an eigenstate with the same energy eigenvalue, which is orthogonal to $\Psi_n(x)$.

(e) [**5 points**] When $a = 0$, the eigenvalue spectrum has no bound states. For $|x| \gg 1$ (so that the potential term $e^{-x^2/2}$ can be neglected), its scattering states are essentially those of a free particle

$$\Psi_p(x) = e^{ipx/\hbar} \begin{pmatrix} \psi_\uparrow \\ \psi_\downarrow \end{pmatrix}.$$

It is paired by Kramers' theorem with $\Psi'_{-p}(x) = \Theta\Psi_p(x)$, which is energetically degenerate with $\Psi_p(x)$ but moving in the opposite direction. Calculate the matrix element $\langle \Psi'_p | e^{-x^2/2} | \Psi_p \rangle$ to show that the back-scattering rate vanishes according to Fermi's Golden rule.

Problem II.5

A particle of charge e and mass m moves in an external magnetic field along the z-direction with magnitude B, in a volume $V = L^3$ with $L \gg \frac{mc}{eB}$.

(a) [**5 points**] Using the gauge $\boldsymbol{A} = (A_x, 0, 0)$, show that the time-independent Schrödinger equation can be written as

$$\frac{\hbar\omega_c}{2}\left[\left(-i\frac{\partial}{\partial x'} + y'\right)^2 - \frac{\partial^2}{\partial y'^2} - \frac{\partial^2}{\partial z'^2}\right]\Psi = E\Psi.$$

where $\omega_c = \frac{eB}{m}$ and the unitless coordinates $x' = x/\ell$, $y' = y/\ell$, and $z' = z/\ell$. Give the "magnetic length" ℓ in terms of ω_c and other quantities in the problem.

(b) [**5 points**] Use the ansatz $\Psi(\boldsymbol{r}) \sim e^{ik_x\ell x'}e^{ik_z\ell z'}\phi(y')$ to write an equation for $\phi(y')$. What is the energy eigenvalue spectrum?

(c) [**5 points**] Use the finite size of the volume to determine the degeneracy of each state with unique eigenenergy.

(d) [**5 points**] Use these levels to evaluate the single-particle partition function Z at high temperature T in the limit where $\hbar\omega_c \ll k_B T$ (k_B is the Boltzmann constant). Retain the lowest-order term dependent on magnetic field.

(e) [**5 points**] Use the partition function to calculate the magnetic susceptibility χ at high temperature. Show that it is diamagnetic for small fields and obeys Curie's law, $\chi \propto T^{-1}$.

Potentially useful:
$\int_0^\infty e^{-x^2}dx = \sqrt{\pi}/2$
$\ln(1+x) \approx x, \quad x \ll 1.$
$\sinh x \approx x + x^3/6, \quad x \ll 1.$