UNIVERSITY OF MARYLAND

Department of Physics

College Park, Maryland

PHYSICS Ph.D. QUALIFYING EXAMINATION

PART II

August 24, 2021 9:00 am – 1:00 pm

Start each problem on a new sheet of paper.

Be sure to write your Qualifier ID ("control number") at the top of each sheet – not your name! – and turn in solutions to four problems only. (If five solutions are turned in, we will only grade #1 - #4.)

At the end of the exam, when you are turning in your papers, please fill in a "no answer" placeholder form for the problem that you skipped, so that the grader for that problem will have something from every student.

You may keep this packet with the questions after the exam.

Problem II.1

Consider a particle of mass m in an asymmetric one-dimensional potential of width a and depth $V_0 > 0$:

$$V(x) = \begin{cases} \infty, & -\infty < x < 0, \\ -V_0, & 0 < x < a, \\ 0, & a < x < \infty. \end{cases} \tag{1}$$

(a) [**9 points**] Derive a transcendental equation that determines the energies E of bound states for this potential.

(b) [**4 points**] What is the minimum depth V_0 for which a bound state exists?

(c) [**4 points**] How many bound states are there for a general depth V_0?

(d) [**4 points**] Suppose the system has a shallow bound state with the energy $E = -0.01\,\hbar^2/2ma^2$. Estimate the probability P to find the particle inside the potential well, with the coordinate $0 < x < a$.

(e) [**4 points**] Suppose the potential (1) has bound states. Now let us modify the potential by adding a positive part outside of the negative part:

$$V(x) = \begin{cases} \infty, & -\infty < x < 0, \\ -V_0, & 0 < x < a, \\ +W_0, & a < x < b, \\ 0, & b < x < \infty, \end{cases} \tag{2}$$

where $W_0 > 0$.

Describe qualitatively how the presence of the positive part modifies energies of the bound states compared with the case $W_0 = 0$. Are they lowered, raised, or unchanged?

Problem II.2

An electron is in the ground state of a hydrogen atom with the energy $E_1 = -\hbar^2/2ma^2$, where a is the Bohr radius and m the electron mass. A weak spatially-uniform time-periodic electric field polarized along the z axis interacts with electron via dipolar coupling

$$\hat{H}_{\text{int}} = -qz\mathcal{E}_0\cos(\omega t), \tag{1}$$

where q and z are the electron charge and coordinate, and $\mathcal{E}_0$ is the electric field amplitude. The frequency ω is such that it may cause ionization of the atom: $\hbar\omega > |E_1|$, i.e., ejection of the electron to an unbound state. Assume that $\hbar\omega - |E_1| \ll mc^2$, so the ejected electron is non-relativistic.

(a) **[2 points]** From conservation of energy, what are the kinetic energy and the momentum magnitude p of the ejected electron? Neglect recoil of the proton, because the proton is much heavier than the electron.

(b) **[7 points]** Using the Fermi golden rule, calculate the rate $d\Gamma/d\Omega$ at which the electron is ejected into a solid angle $d\Omega$ about the direction of its momentum $\boldsymbol{p}$.

Express your result in terms of a matrix element $M_{\boldsymbol{p}0}$ (to be calculated in Part (c)) of the perturbation (1) between the initial and final states of the electron.

(c) **[7 points]** Now calculate the matrix element $M_{\boldsymbol{p}0}$ of the perturbation (1) between the ground state $\psi_0(r)$ of the electron and its unbound state $\psi_{\boldsymbol{p}}(\boldsymbol{r})$ approximated as a plane wave of momentum $\boldsymbol{p}$.

(d) **[3 points]** Substitute the result of Part (c) into Part (b) and obtain an explicit formula for the ejection rate $d\Gamma/d\Omega$.

How does $d\Gamma/d\Omega$ depend on the angle θ between the electron momentum $\boldsymbol{p}$ and the electric field polarized along z? At which angles is it maximal and minimal?

(e) **[3 points]** Verify that $d\Gamma/d\Omega$ obtained in Part (d) has the correct dimension of 1/Time.

(f) **[3 points]** Express $d\Gamma/d\Omega$ in terms of the dimensionless ratio $\hbar\omega/|E_1|$ by eliminating p using Part (a). Sketch and describe how $d\Gamma/d\Omega$ depends on frequency ω.

Possibly useful information:

The ground-state wave function of the hydrogen atom is $\psi_0(r) = e^{-r/a}/\sqrt{\pi a^3}$.

The following integral may be useful:

$$I(\boldsymbol{p}) = \int d^3r\, e^{-i\boldsymbol{p}\cdot\boldsymbol{r}/\hbar}\, e^{-r/a} = \frac{8\pi a^3}{[1+(pa/\hbar)^2]^2}. \tag{2}$$

Another useful integral can be deduced from the integral (2):

$$\int d^3r\, z\, e^{-i\boldsymbol{p}\cdot\boldsymbol{r}/\hbar}\, e^{-r/a} = i\hbar\frac{dI(\boldsymbol{p})}{dp_z}. \tag{3}$$

Problem II.3

A hypothetical spinless electrically-neutral particle of mass m interacts (only) with the electrons via the contact potential

$$U(\boldsymbol{r} - \boldsymbol{r}_e) = \Lambda\, \delta^{(3)}(\boldsymbol{r} - \boldsymbol{r}_e), \tag{1}$$

where $\boldsymbol{r}$ and $\boldsymbol{r}_e$ are the coordinates of the particle and the electron, and $\delta^{(3)}$ is the three-dimensional Dirac's delta-function.

Consider elastic scattering of the particle with a non-relativistic energy $E = \hbar^2 k^2/2m$ on a hydrogen atom, where the electron is in the ground state and remains in the ground state after the particle has scattered.

(a) **[3 points]** Calculate the effective scattering potential $V(r)$ experienced by the particle

$$V(r) = \int d^3 r_e\, U(\boldsymbol{r} - \boldsymbol{r}_e)\, |\psi_0(\boldsymbol{r}_e)|^2 \tag{2}$$

in terms of the electron wave function $\psi_0(r_e) = e^{-r_e/a}/\sqrt{\pi a^3}$ of the ground state of the hydrogen atom and the Bohr radius a.

(b) **[8 points]** In the first Born approximation, calculate the scattering amplitude $f(\theta)$ of the particle.

Then, using this $f(\theta)$, compute the differential cross section $d\sigma/d\Omega$ of scattering.

What is the ratio $f(\theta = \pi)/f(\theta = 0)$?

At which scattering angles θ is $d\sigma/d\Omega$ maximal and minimal?

(c) **[8 points]** Using the result of Part (b), calculate the total cross section σ of scattering. Express the final answer for σ in terms of the kinetic energy E of the particle.

From your result, obtain $\sigma(E = 0)$ in the low-energy limit $E \to 0$.

Also obtain an asymptotic formula for $\sigma(E)$ in the high-energy limit $E \gg \hbar^2/ma^2$.

(d) **[6 points]** Formulate conditions of applicability of the Born approximation in the low- and high-energy limits, in terms of the interaction strength Λ and the speed v of the particle.

Hint: Compare σ with the geometrical area $\sim a^2$ of the scattering potential.

Possibly useful integral:

$$I(\boldsymbol{q}) = \int d^3 r\, e^{i\boldsymbol{q}\cdot\boldsymbol{r}}\, e^{-2r/a} = \frac{\pi a^3}{(1 + q^2 a^2/4)^2}. \tag{3}$$

Problem II.4

Two non-relativistic *identical* fermions of mass m and spin $1/2$ interact via the potential

$$V = -\frac{g}{r}\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2,$$

where $g > 0$, r is the distance between the particles, and $\boldsymbol{\sigma}_j$ are the Pauli spin matrices operating on the state of particle j.

(a) **[5 points]** Write the eigenstates of $\boldsymbol{S}^2_{\text{tot}} \equiv \boldsymbol{S}_{\text{tot}} \cdot \boldsymbol{S}_{\text{tot}}$, where $\boldsymbol{S}_{\text{tot}} = \boldsymbol{S}_1 + \boldsymbol{S}_2$ is the total spin operator, in terms of products of single spin-1/2 states, and give the corresponding eigenvalues. (A derivation is not required.)

(b) **[3 points]** Show that the total spin $\boldsymbol{S}_{\text{tot}}$ commutes with V, and explain why the energy eigenstates can be taken to have definite total spin S.

(c) **[3 points]** Show that the potential is attractive for some spin states, and identify those spin states.

(d) **[2 points]** Give an argument showing that the two-particle system has bound states.

(e) **[2 points]** Explain why the energy eigenstates can be taken to have definite total orbital angular momentum L, in addition to having definite total S.

(f) **[5 points]** What are S and L in the ground state of this system of two identical fermions? Justify your answer.

(g) **[5 points]** What are the energy and the degeneracy of the ground state?

Possibly useful information: The energy levels of the hydrogen atom are given by

$$E_n = -\frac{m_e e^4}{2n^2\hbar^2}$$

in the limit of $m_p \gg m_e$, where $m_{e,p}$ are the electron and proton masses, respectively.

Problem II.5

This problem deals with the first quantum mechanical model that can account for the observation that the heat capacity per volume $C_V(T)$ of insulators is much smaller at low temperatures than the classical result (the Dulong-Petit Law).

(a) **[4 points]** Derive an expression for the average **energy** at temperature T of a **quantum** harmonic oscillator of the angular frequency ω in **one dimension** $(D = 1)$.

(b) **[4 points]** Mindful of Planck's results and the quantum theory of oscillators, Albert Einstein proposed a crude model of insulators. He set all the vibrational modes of the N atoms in a three-dimensional ($D = 3$) solid insulator of volume V to have the same frequency ω_E. Find $C_V(T)$ of this so-called Einstein model.

(c) **[2 points]** How would $C_V(T)$ change if the problem were formulated in one- or two-dimensional space?

(d) **[3 points]** Find the high-temperature limit of C_V in $D = 3$ and verify that it agrees with the classical result coming from the Equipartition Theorem.

(e) **[3 points]** Find the expression to which $C_V(T)$ simplifies at temperatures well below $\hbar\omega_E/k_B$, and then evaluate $C_V(0)$.

(f) **[2 points]** Sketch the behavior of $C_V(T)$ vs. T from $T = 0$ to a temperature a few times $\hbar\omega_E/k_B$.

(g) **[2 points]** Phonon modes in a solid can be acoustic or optical. Which of these modes are better described by the Einstein model?

(h) **[3 points]** Why does the Einstein model poorly describe the magnitude and thermal behavior of $C_V(T)$ of a metal?

(i) **[2 points]** Within the Einstein model, compare two single-element insulators, both having the same N and V but with each made entirely of one of two different isotopes of the element. How, if at all, would $C_V(T)$ differ? In the limits of high and of low temperature, would C_V be larger or smaller for the insulator with the higher-mass isotope?