UNIVERSITY OF MARYLAND
Department of Physics
College Park, Maryland

PHYSICS Ph.D. QUALIFYING EXAMINATION
PART II

August 23, 2018 9:00 a.m. – 1:00 p.m.

Do any four problems. Each problem is worth 25 points.
Start each problem on a new sheet of paper (because different faculty members will be grading each problem in parallel).

Be sure to write your Qualifier ID ("control number") at the top of each sheet — not your name! — and turn in solutions to four problems only. (If five solutions are turned in, we will only grade # 1 - # 4.)

At the end of the exam, when you are turning in your papers, please fill in a "no answer" placeholder form for the problem that you skipped, so that the grader for that problem will have something from every student.

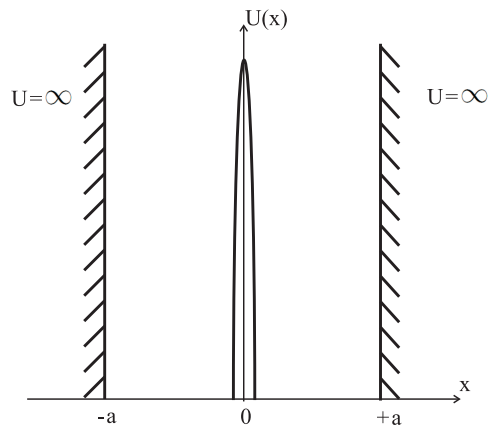
You may keep this packet with the questions after the exam.

Problem II.1

Consider a quantum particle of mass m confined within a one-dimensional infinite square-well potential in the presence of an additional repulsive delta-potential in the middle of the well:

$$U(x) = \begin{cases} \beta\,\delta(x), & \text{if } -a < x < a; \\ \infty, & \text{if } |x| > a, \end{cases} \tag{1}$$

where $\delta(x)$ is the Dirac δ-function and $\beta \geq 0$. The potential $U(x)$ is illustrated below.



(a) [**2 points**] Consider the parity (reflection) operator $\hat{P}$, whose action on an arbitrary function $\psi(x)$ is defined as follows:

$$\hat{P}\,\psi(x) = \psi(-x)\,.$$

Explain (briefly) why the eigenstates of the potential $U(x)$, given by Eq. (1), should also be eigenstates of the parity operator.

(b) [**4 points**] Consider the stationary Schrödinger equation with the energy E corresponding to the potential (1). What boundary conditions must the wave function satisfy at $x = \pm a$? Also, prove that the wave function satisfies the following matching condition at $x = 0$:

$$\psi'(0^+) - \psi'(0^-) = \frac{2m\beta}{\hbar^2}\psi(0), \tag{2}$$

where the left-hand side represents the difference of the derivatives $\psi' = d\psi/dx$ taken in the limit $x \to 0$ from the positive side ($x > 0$) and from the negative side ($x < 0$).

(c) [**3 points**] Show that the odd-parity ($-$) and even-parity ($+$) wave function forms below both satisfy the Schrödinger equation everywhere that the potential is finite, and determine a relationship between the parameters $k_\pm$ and the energy E.

$$\psi_-(x) = \sin\left[k_-\,x\right] \tag{3}$$

$$\psi_+(x) = \sin\left[k_+\,(|x| - a)\right] \tag{4}$$

(d) [**4 points**] Apply the boundary conditions formulated above to the odd-parity wave function $\psi_-(x)$ from Eq. (3) and determine the allowed values of k_- and the corresponding eigenenergies E_-. How are the eigenenergies E_- affected by the presence of the δ-function potential in the middle of the well?

(e) [**4 points**] Prove that the even-parity wave function $\psi_+(x)$ from Eq. (4) is an eigenfunction of the Schrödinger equation if the wave vector k_+ satisfies the following equation:

$$\tan(k_+ a) = -\frac{k_+ a}{\xi}, \quad \text{where} \quad \xi = \frac{ma\beta}{\hbar^2}. \tag{5}$$

From Eqs. (1) and (5), determine the dimensionality of β and ξ.

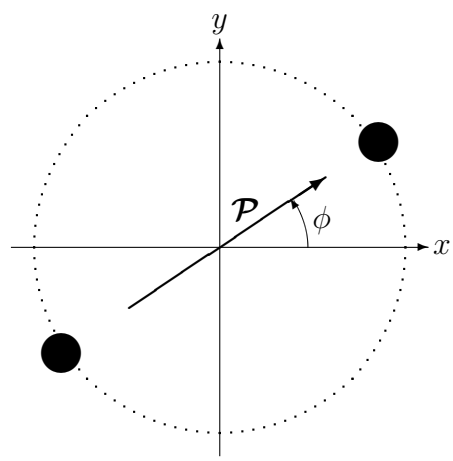
(f) [**4 points**] Examine Eq. (5) in the limit $\xi \to 0$, which corresponds to a vanishingly weak δ-function potential. Determine the allowed values of k_+ and the corresponding even-parity energy levels E_+ in this case. Compare your result with the well-known spectrum of an infinite potential well without δ-function potential ($\beta = 0$).

(g) [**4 points**] Now consider the limit of a very strong δ-function potential: $\xi \to \infty$. Determine the allowed values of k_+ and the corresponding energy levels E_+ from Eq. (5) in this limit. How does this energy spectrum of even-parity eigenstates compare with the energy spectrum of the odd-parity eigenstates found in part (d)?

Problem II.2

A diatomic molecule with moment of inertia I is constrained to rotate freely in the xy plane with angular momentum L_z. The molecule has a permanent electric dipole moment $\boldsymbol{\mathcal{P}}$ along the molecule axis, whose magnitude $\mathcal{P}_0$ is independent of the rotational motion or external conditions. The Hamiltonian for the quantum system is

$$\hat{H} = \frac{(\hat{L}_z)^2}{2I} \,. \tag{1}$$



(a) [**6 points**] The orientation of the dipole moment $\boldsymbol{\mathcal{P}}$ relative to the x axis is specified by the angle ϕ as shown in the figure. Write down the Hamiltonian (1) in the ϕ representation and obtain the energy eigenvalues E_n and eigenfunctions $\psi_n(\phi)$. What is the degeneracy of each eigenstate?

(b) [**5 points**] Now a weak uniform electric field $\boldsymbol{\mathcal{E}} = \mathcal{E}_0\, \hat{e}_x$ is applied to the system along the x axis, adding a term $-\boldsymbol{\mathcal{P}}\cdot\boldsymbol{\mathcal{E}}$ to the Hamiltonian. Write down the new Hamiltonian for the system in terms of ϕ, then use perturbation theory to calculate the shifts of the energy eigenvalues to first order in $\mathcal{E}_0$. Does the perturbation lift the degeneracy to first order in $\mathcal{E}_0$?

(c) [**6 points**] Use perturbation theory to calculate the corresponding perturbed wave functions to first order in $\mathcal{E}_0$.

(d) [**5 points**] Evaluate the expectation value $\langle \mathcal{P}_x \rangle$ of the x component of the dipole moment operator $\boldsymbol{\mathcal{P}}$ in each energy eigenstate; denote the expectation value for state ψ_n by $\langle \mathcal{P}_x \rangle_n$. From that, deduce the electric polarizability

$$\alpha_n = \langle \mathcal{P}_x \rangle_n \,/\, \mathcal{E}_0 \tag{2}$$

of each state.

(e) [**3 points**] Provide a physical explanation for the difference in the signs of α for the lowest energy eigenstate versus the other states. (Hint: Classically, would the dipole spend more time aligned or anti-aligned with the applied field?)

Problem II.3

The differential cross-section $d\sigma/d\Omega$ for the elastic scattering of a particle of mass m from a target characterized by a potential $V(\boldsymbol{r})$ is

$$\frac{d\sigma}{d\Omega} = |f(\boldsymbol{k}, \boldsymbol{k}')|^2, \tag{1}$$

where $\hbar\boldsymbol{k}$ and $\hbar\boldsymbol{k}'$ are the incident and scattered momenta of the particle, with $|\boldsymbol{k}| = |\boldsymbol{k}'|$. In the first-order Born approximation, the scattering amplitude $f(\boldsymbol{k}, \boldsymbol{k}')$ can be written as

$$f(\boldsymbol{k}, \boldsymbol{k}') = -\frac{1}{4\pi}\frac{2m}{\hbar^2}\int d^3r\; e^{i(\boldsymbol{k}-\boldsymbol{k}')\cdot\boldsymbol{r}}\; V(\boldsymbol{r})\ . \tag{2}$$

The first-order Born approximation is valid if

$$\left|\frac{2m}{\hbar^2}\frac{1}{4\pi}\int d^3r\;\frac{e^{ikr}}{r}V(\mathbf{r})e^{i\mathbf{k}\cdot\mathbf{r}}\right| \ll 1\ . \tag{3}$$

(a) [**2 points**] Show that $q \equiv |\boldsymbol{k} - \boldsymbol{k}'|$ can be written as

$$q = 2|\boldsymbol{k}|\sin(\theta/2), \tag{4}$$

where θ is the angle between $\boldsymbol{k}$ and $\boldsymbol{k}'$ (the scattering angle).

(b) [**5 points**] For a spherically symmetric potential $V(r)$, show that Eq. (2) reduces to

$$f(\theta) = -\frac{2m}{\hbar^2}\frac{1}{q}\int_0^\infty dr\, r\, V(r)\,\sin(qr) \tag{5}$$

expressed using the direct correspondence between q and θ.

(c) [**5 points**] Let $V(r)$ be the Yukawa potential

$$V(r) = \frac{V_0}{\mu}\frac{e^{-\mu r}}{r}. \tag{6}$$

Show that in this case, the scattering amplitude is

$$f(\theta) = \frac{2mV_0}{\hbar^2\mu}\frac{1}{q^2+\mu^2}. \tag{7}$$

(d) [**5 points**] Recall the Coulomb potential

$$V_C(r) = \frac{ZZ'e^2}{4\pi r}, \tag{8}$$

and the Rutherford formula for Coulomb scattering

$$\frac{d\sigma_C}{d\Omega} = \frac{4m^2Z^2Z'^2e^4}{\hbar^4}\frac{1}{16\,k^4\sin^4(\theta/2)}, \tag{9}$$

where Ze is the electric charge of the scattered particle and $Z'e$ is the electric charge of the target. Find the appropriate limit on μ and V_0/μ such that the Yukawa potential (6) reduces to the Coulomb potential and show that the scattering amplitude (7) produces the Rutherford formula in this limit.

(e) [**4 points**] Consider the Yukawa potential (6) and very low scattering momenta such that $|\mathbf{k}| \ll \mu$. Under what condition (in terms of V_0, μ and m) is the first-order Born approximation valid?

(f) [**4 points**] Consider the Yukawa potential (6) and very high scattering momenta such that $|\mathbf{k}| \gg \mu$. Under what condition (in terms of V_0, μ, m and $|\mathbf{k}|$) is the first-order Born approximation valid?

Possibly useful:

$\int_{-1}^{1} dx \int_0^\infty e^{iar(1+x)-br} = \frac{1}{a}\log(1 - \frac{2ia}{b})$ for real and positive a and b

$\log(1+x) \approx x$ for $x \ll 1$

$\log(1+x) \approx \log(x)$ for $x \gg 1$

Problem II.4

Consider a system consisting of two particles, A and B, each of spin 1. The two particles may form a bound state, which we treat as a composite particle C. The states of the particles are represented in the basis $|j, m\rangle_{A,B,C}$, where j and m are the quantum numbers of the angular momentum and its projection on the z axis, and the index A, B, or C indicates the particle. In this problem, assume conservation of angular momentum. Consider only the spin angular momentum of the particles and ignore the orbital angular momentum and the spatial part of the wave functions.

The particles A and B are bosons. Consider two cases: (i) the particles A and B are distinguishable (i.e. different), (ii) the particles A and B are indistinguishable (i.e. identical).

(a) [**5 points**] Given that $j_A = 1$ and $j_B = 1$, what are the possible values of the angular momentum j_C of the composite particle C? Answer the question in the two cases (i) and (ii). In the latter case, discuss the symmetry of the composite wave function with respect to interchange of the particles A and B to argue that one certain j_C value is forbidden.

(b) [**5 points**] Suppose the Hamiltonian of the system is

$$\hat{H} = a\,(\hat{J}_{z,C})^2,$$

where $\hat{J}_{z,C} = \hat{J}_{z,A} + \hat{J}_{z,B}$ is the z component of the angular momentum operator of the composite particle C, and a is a coefficient.

What are the eigenvalues of the Hamiltonian, and what are their degeneracies? Answer the question in the two cases (i) and (ii).

(c) [**6 points**] Suppose initially one of the particles A and B is in the state $|1, 1\rangle$ and another in the state $|1, -1\rangle$. Then, these particles combine to form the particle C. What are the possible states $|j, m\rangle_C$ of the composite particle C in this case? What are the probabilities of finding the particle C in these states?

Answer the question in the two cases (i) and (ii). In the latter case, write down the properly symmetrized wavefunction of the initial state of the particles A and B.

(d) [**5 points**] Suppose the particle C is in the state $|0, 0\rangle_C$. Suppose it is a metastable state, and the particle C decays to particles A and B. Write down the wavefunction of the two-particle system $|\psi\rangle_{AB}$ in the basis of the states $|1, m_1\rangle_A |1, m_2\rangle_B$. For each case [(i) and (ii)], what are the permitted combinations of the numbers m_1 and m_2, and what are the probabilities of finding the particles A and B in these states?

(e) [**4 points**] Considering the composite particle C from the previous part, suppose that a measurement finds the particle A in the state $|1, 1\rangle_A$. Then, what are the possible states $|1, m\rangle_B$ of the particle B? Justify your answer.

Information about selected Clebsch-Gordan coefficients is below:

$$|1,1\rangle|1,-1\rangle = \sqrt{\frac{1}{6}}\,|2,0\rangle + \sqrt{\frac{1}{2}}\,|1,0\rangle + \sqrt{\frac{1}{3}}\,|0,0\rangle \tag{1}$$

$$|1,0\rangle|1,0\rangle = \sqrt{\frac{2}{3}}\,|2,0\rangle \qquad\qquad - \sqrt{\frac{1}{3}}\,|0,0\rangle \tag{2}$$

$$|1,-1\rangle|1,1\rangle = \sqrt{\frac{1}{6}}\,|2,0\rangle - \sqrt{\frac{1}{2}}\,|1,0\rangle + \sqrt{\frac{1}{3}}\,|0,0\rangle \tag{3}$$

$$|2,0\rangle = \sqrt{\frac{1}{6}}\,|1,1\rangle|1,-1\rangle + \sqrt{\frac{2}{3}}\,|1,0\rangle|1,0\rangle + \sqrt{\frac{1}{6}}\,|1,-1\rangle|1,1\rangle \tag{4}$$

$$|1,0\rangle = \sqrt{\frac{1}{2}}\,|1,1\rangle|1,-1\rangle \qquad\qquad - \sqrt{\frac{1}{2}}\,|1,-1\rangle|1,1\rangle \tag{5}$$

$$|0,0\rangle = \sqrt{\frac{1}{3}}\,|1,1\rangle|1,-1\rangle - \sqrt{\frac{1}{3}}\,|1,0\rangle|1,0\rangle + \sqrt{\frac{1}{3}}\,|1,-1\rangle|1,1\rangle \tag{6}$$

Problem II.5

In this problem you will compare noninteracting boson gases in three dimensions ($d = 3$) and in two dimensions ($d = 2$). Some interesting statistical properties are qualitatively different depending on the dimensionality.
For both cases ($d = 3, 2$), consider N noninteracting spinless bosons of mass m in a volume $V = L^d$. Assume periodic boundary conditions so that the single-particle energy levels are eigenstates of momentum $\hbar\mathbf{k}$ with energy $\epsilon(k) = \hbar^2 k^2/2m$. Also assume that the system size is large so that it is safe to calculate in the continuous (not discrete) limit.

(a) [**6 points**] Find the density of states as a function of energy, $\nu_d(\epsilon)$. (Hint: first calculate the total number of states with energy $\leq \epsilon$, then determine the density of states from that.)

(b) [**2 points**] Having found $\nu_3(\epsilon)$ and $\nu_2(\epsilon)$, what is the qualitative difference between them as $\epsilon \to 0$?

(c) [**2 points**] Now assume a grand canonical ensemble with a given *average* number density of particles, $\langle N \rangle / V$. For an arbitrary d, write down an expression equating the average number density, $n \equiv \langle N \rangle / V$, to an integral that involves $\nu_d(\epsilon)$ and the Bose-Einstein distribution function

$$n_{BE}(\epsilon) = \frac{1}{e^{(\epsilon-\mu)/k_B T} - 1}, \tag{1}$$

where μ the chemical potential. (Do not try to evaluate the integral at this time.)

(d) [**3 points**] We assert (without asking you to prove it here) that μ must be < 0. Assuming that the temperature is fixed, explain how you can tell from the integral that $n(\mu)$ increases as $\mu \to 0^-$.

(e) [**6 points**] For $d = 3$, show that at fixed temperature T, $n(\mu)$ increases to a finite value as $\mu \to 0^-$. Invert the relationship to obtain an expression for the critical temperature, T_c, as a function of n.

(f) [**2 points**] For a $d = 3$ Bose gas, what happens when $n > n_c$ for a given T, or $T < T_c$ for a given n? What is the experimental signature in the momentum distribution when the critical value is exceeded?

(g) [**4 points**] For $d = 2$, prove that the integral relation for n can increase without bound as $\mu \to 0^-$. That means that there is no critical temperature in the two-dimensional Bose gas—an important qualitative difference from the three-dimensional case.

Possibly useful:

$$\zeta(3/2) = \frac{4}{\sqrt{\pi}} \int_0^\infty dx \frac{x^2}{\exp(x^2) - 1} = 2.612...$$