UNIVERSITY OF MARYLAND
Department of Physics
College Park, Maryland

PHYSICS Ph.D. QUALIFYING EXAMINATION
PART I

August 24, 2017 **9:00 a.m. – 1:00 p.m.**

Do any four problems. Each problem is worth 25 points.
Start each problem on a new sheet of paper (because different faculty members will be grading each problem in parallel).

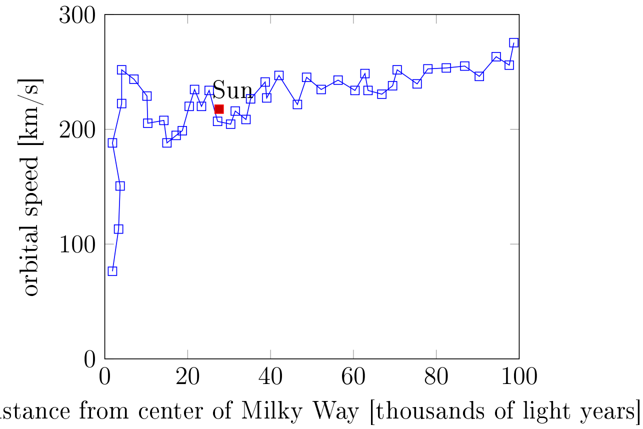
Be sure to write your Qualifier ID ("control number") at the top of each sheet — not your name! — and turn in solutions to four problems only. (If five solutions are turned in, we will only grade # 1 - # 4.)

At the end of the exam, when you are turning in your papers, please fill in a "no answer" placeholder form for the problem that you skipped, so that the grader for that problem will have something from every student.

You may keep this packet with the questions after the exam.

Problem I.1

The rotation curve of a galaxy is a plot of the orbital speeds of visible stars versus their radial distance from the galactic center. This is shown below for our Milky Way galaxy.



(a) **[5 points]** According to Newton/Kepler's law, the orbital speed of a star in the outer parts of the galaxy decreases with distance r from the galactic center. Derive a formula that describes this. You may assume that the star is sufficiently far out so that the total mass of the galaxy can be taken to be a constant point mass M.

(b) **[5 points]** The rotation curve, however, shows a fairly flat speed distribution. Suppose we assume there is a spherically symmetric invisible mass distribution inside and outside the visible galaxy. Let the total mass density be $\rho(r)$. For a star in Keplerian orbit inside such mass distribution, deduce how $\rho(r)$ must vary with r if the speed distribution is assumed constant. How does $M(r)$ vary with r?

(c) **[5 points]** Observations indicate that the density of luminous (visible) matter in our galaxy decreases with distance as $\rho_L(r) \sim r^{-3.5}$. How does this compare with the mass density distribution in (b), and what does this imply for the distributions of visible matter and invisible (dark) matter, especially at large r?

(d) **[5 points]** There is evidence, from dwarf galaxies, etc, that the flat speed distribution persists up to 300,000 light years, or, about six times the radius of the Milky Way. If so, what are the implications for the relative mass ratio between visible and dark matter? [Refer to your results in part (b).]

(e) **[5 points]** In one approach, referred to as Modified Newtonian Dynamics, Newton's second law is modified to $F = ma^2/a_0$, for small accelerations $a << a_0$, where a_0 is a constant. Show how this hypothesis could "explain" the constant rotation curve described above. Given that there is a relatively firm basis to expect Newton's equation to be a vector relationship, how would you assess this approach?

Problem I.2

(a) [**4 points**] A charge $+q$ is located at $(x, y, z) = (0, 0, h)$ above an infinite grounded conducting plate located in the $x-y$ plane at $z = 0$. Calculate the electric potential for $z > 0$ by the method of images. Show clearly that your solution satisfies the boundary condition of zero tangential $\boldsymbol{E}$ field at the plate. What is the force on the charge $+q$ due to induced charges on the conducting plate? [Use $k = 1/(4\pi\epsilon_0)$.]

(b) [**5 points**] Suppose we replace the $+q$ charge with a permanent point dipole $\boldsymbol{p} = p_0\hat{\boldsymbol{z}}$. By modeling the dipole as two opposite closely spaced charges, deduce which way the image dipole points. Using the formulae provided below, calculate the electric potential of the system for $z > 0$. Show clearly that your solution satisfies the boundary condition of zero tangential $\boldsymbol{E}$ field at the plate, and that it is consistent with your deduced dipole directions.

(c) [**3 points**] Calculate the $\boldsymbol{E}$ field at $z = h$ due to the image dipole $\boldsymbol{p}_0$. Which way is the force between the dipoles?

(d) [**5 points**] Suppose we replace the $+q$ charge from part (a) with a spherical, neutral conductor of radius a. In what follows, assume $a << h$. You are reminded that, in the presence of any applied electric field, the spherical conductor will develop a dipole moment $\boldsymbol{p} = \boldsymbol{E}_{\text{applied}} a^3/k$. Suppose such a dipole moment is formed, in the $\hat{z}$ direction, as an initial fluctuation. From your results (b) and (c), calculate the ratio of the E field (at $z = h$) from the induced dipole to the fluctuation E_{applied} field?.

(e) [**4 points**] Discuss the stability of this situation: under what condition on h is the initial fluctuation of the E_{applied} field self-consistent with the E field from the induced charges on the conducting plate? If this condition is obtained, what will be the subsequent motion of the sphere? Is the condition on h you obtain realistic given the assumption $a/h << 1$?

(f) [**4 points**] Suppose h is large so that your initial fluctuation is not self-consistent. *Qualitatively*, state briefly what will be the subsequent evolution of the system; in particular, will the sphere move significantly?; and/or, how will the initial charge distribution evolve?

Potentially useful: The electric potential due to a point dipole $\boldsymbol{p}$, at position vector $\boldsymbol{r}$ from the dipole center, is given by $\phi(\boldsymbol{r}) = k\boldsymbol{p} \cdot \boldsymbol{r}/r^3$. The electric field resulting from this potential is $\boldsymbol{E}(\boldsymbol{r}) = (k/r^3)[3\hat{\boldsymbol{r}}(\boldsymbol{p} \cdot \hat{r}) - \boldsymbol{p}]$.

Problem I.3

For this problem a chain is defined as a classical, one-dimensional extended object which lives on the links of a three dimensional cubic lattice with link length a, and has energy per unit length ϵ. We will ignore any interaction of the chain with itself (i.e. there is no extra energy if the chain crosses itself, or if two segments of the chain lie on the same link of the lattice).

Suppose that one end of the chain is fixed at the origin, but that the number of links and the position of the other end can fluctuate.

(a) [**5+5 points**] Assuming the chain is in thermal equilibrium at temperature T, evaluate (i) its partition function, and (ii) its average length.

(b) [**5 points**] The chain cannot be in equilibrium above some temperature T_c. Find an expression for T_c, and describe what happens to the chain as T approaches it from below.

Now suppose that the number of links in the chain is fixed, and that both ends are fixed, at a distance L from each other. Let the number of such configurations be denoted by $g(L)$. You need not evaluate $g(L)$, and your answers may refer to $g(L)$.

(c) [**5 points**] Assuming the chain is in thermal equilibrium at temperature T, what is the entropy of the chain?

(d) [**5 points**] Using thermodynamic reasoning, and treating L as a continuous state parameter, find the magnitude and direction of the force exerted by the chain on the anchors at its ends. (*Hint*: The internal energy of the chain is determined only by the number of links, so is independent of the distance L and the temperature.)

Problem I.4

At $t = 0$ a particle of rest mass m, charge e, is placed in a uniform electric field $\boldsymbol{E} = E_0 \hat{\boldsymbol{y}}$ with an initial velocity $\boldsymbol{v} = v_0 \hat{\boldsymbol{x}}$, where v_0 may be comparable to c.

(a) [**6 points**] Calculate for $t > 0$ the time dependence of the relativistic particle's momenta $\boldsymbol{p} = (p_x, p_y)$, and energy. [Use standard definitions $\beta = v/c$ and $\gamma^2 = (1-\beta^2)^{-1}$.]

(b) [**7 points**] From the results of (a), calculate the particle velocities $v_x(t)$ and $v_y(t)$. Identify a characteristic time τ such that for $t \ll \tau$ the particle energy is very close to the initial relativistic mass energy and for $t \gg \tau$ the energy much exceeds this initial energy. Write an expression for τ in terms of the parameters of the problem, (e, m, E_0, v_0, c), and rewrite the velocities in terms of τ, v_0, and c. Make a sketch of the velocities as a function of time. What are (v_x, v_y) for $t \to \infty$?

(c) [**7 points**] Solve for $x(t)$ and $y(t)$ if $x(0) = 0$, $y(0) = 0$. Derive an expression for the trajectory $y(x)$.

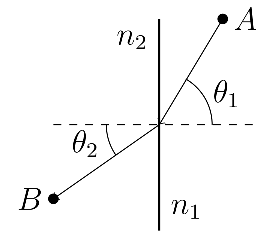
Now consider this problem from the viewpoint of the reference frame in which the particle is initially at rest.

(d) [**5 points**] Based on what you know about the motion in the lab frame, what is the velocity of the particles as $t' \to \infty$ in the transformed frame?

Potentially useful:

$$\int \frac{ds}{(1+s^2)^{1/2}} = \sinh^{-1}(s), \qquad \cosh^2(s) - \sinh^2(s) = 1.$$

Problem I.5



Fermat's principle states that the path of a light ray between points A and B through media with varying index of refraction $n(x)$ minimizes the total travel time t.

(a) [**8 points**] Use Fermat's principle to derive Snell's law at a single interface of dissimilar materials, which relates the refraction indices n_1 and n_2 and normal incidence angles θ_1 and θ_2.

If we write the travel time as a path integral

$$t = \int_{\ell_A}^{\ell_B} \frac{d\ell}{v(\ell)}, \tag{1}$$

we can cast Fermat's principle in terms of variational calculus.

(b) [**4 points**] Show that Eq. (1) can be written as

$$t = \frac{1}{c} \int n(x,y)\sqrt{1+y'^2}\ dx. \tag{2}$$

where $y' = \frac{dy}{dx}$ and c is the speed of light in vacuum.

(c) [**8 points**] Use the Euler-Lagrange equation associated with Eq. (2) to show that when $n(x,y) \to n(x)$, the optimal path conserves the quantity

$$\frac{n(x)y'}{\sqrt{1+y'^2}}. \tag{3}$$

(d) [**5 points**] Show that Eq. (3) is equivalent to Snell's law.