UNIVERSITY OF MARYLAND
Department of Physics
College Park, Maryland

PHYSICS Ph.D. QUALIFYING EXAMINATION
PART II

August 25, 2017 **9:00 a.m. – 1:00 p.m.**

Do any four problems. Each problem is worth 25 points.
Start each problem on a new sheet of paper (because different faculty members will be grading each problem in parallel).

Be sure to write your Qualifier ID ("control number") at the top of each sheet — not your name! — and turn in solutions to four problems only. (If five solutions are turned in, we will only grade # 1 - # 4.)

At the end of the exam, when you are turning in your papers, please fill in a "no answer" placeholder form for the problem that you skipped, so that the grader for that problem will have something from every student.

You may keep this packet with the questions after the exam.

Problem II.1

Consider a free electron (non-relativistic) of mass m and charge e in three-dimensions, with Cartesian coordinates (x, y, z). The electron is subject to a magnetic field in the z direction, $\boldsymbol{B} = B\hat{\boldsymbol{z}}$ and a potential $V(z) = \infty$ for $|z| > a$ and $V(z) = 0$ for $|z| \leq a$.

(a) [**10 points**] Write down the Hamiltonian in Landau gauge, $\boldsymbol{A} = (0, Bx, 0)$. Show that the Schrödinger equation separates, and that the energy eigenstates are determined by wave functions of the form $\Psi_{k_y,n,n'}(x, y, z) = e^{ik_y y}\chi_n(x)\phi_{n'}(z)$, where n and n' are non-negative integers. Write down the energies of the eigenstates in terms of k_y, n, and n'.

(b) [**5 points**] Sketch the wave functions $\phi_{n'}(z)$ for $n' = 0, 1, 2$. Discuss conditions under which we can ignore the states with $n' > 0$.

Consider now the case of a relativistic particle of charge e moving in the two-dimensional $x - y$ plane, subject to the same perpendicular magnetic field as above. The system is described by the Hamiltonian:

$$H = \sigma_x v(p_x + eA_x) + \sigma_y v(p_y + eA_y), \tag{1}$$

where $\boldsymbol{p} = \frac{\hbar}{i}\boldsymbol{\nabla}$ are the momenta, $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ are the Pauli matrices, and v is the speed of the particle. This is an effective Hamiltonian for valence electrons in graphene.

(c) [**10 points**] What are the energy levels of this system? [Hint: Write the wave function as $\Psi = \begin{pmatrix} \Phi_\mathrm{A} \\ \Phi_\mathrm{B} \end{pmatrix}$, and first solve for the square of the energy levels. The results of part (a) above may be useful.].

Problem II.2

Consider a particle of mass m moving in a spherically symmetric potential

$$V(r) = \begin{cases} -\frac{\alpha}{r} & \text{if } r > R \\ -\frac{\alpha}{R} & \text{if } r < R \end{cases}$$

(a) [**4 points**] Assuming $V(r)$ is due to electrostatic interaction with the particle, describe the distribution of charge that would give this potential.

(b) [**8 points**] The ground state wave function of the particle moving in the *pure Coulomb potential*

$$V_0(r) = -\frac{\alpha}{r}$$

is

$$\Psi_0 = A\exp(-r/a).$$

Express A, a, and the energy of the state, E, in terms of m, α, and $\hbar$.

(c) [**6 points**] Consider the potential given in part (a) as a perturbation about the pure Coulomb potential, and use first order perturbation theory to find the energy shift of the ground state normalized to the ground state energy. *There will be a radial integral that you may leave unevaluated.*

(d) [**7 points**] We will use asymptotic methods to do the above integral. Note, first, that the exponential function controls the size of the integrand for different values of r/a.

(i) Consider, then, the limit $R << a$. By examining the domain of the integration, and then appropriately approximating the exponential in that domain, evaluate the integral in this limit. What is the relative energy shift? Is perturbation theory valid?

(ii) Now, consider $R >> a$. Physically, in this limit, do you expect perturbation theory to work? Explain. Regardless, by examining the size of the integrand for large r, and then appropriately extending the upper bound of the integration, estimate the integral (possibly using the integrals provided). Comment on the energy shift, and the validity of perturbation theory in this limit.

Potentially useful:

$$\nabla^2 = \frac{1}{r^2}\left(\frac{\partial}{\partial r}\right)\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}$$

$$\int dx\, x\, e^{-x} = -\,(x+1)\, e^{-x}$$

$$\int dx\, x^2 e^{-x} = -\left(x^2+2x+2\right) e^{-x}$$

Problem II.3

Consider **inelastic** scattering of light particles of mass M and electric charge e (say, muons) on the hydrogen atom. The muons have high initial velocity v_i, such that

$$v_i \gg \frac{e^2}{\hbar}, \quad \frac{\hbar}{ma}, \quad \frac{\hbar}{Ma}, \tag{1}$$

where m is the electron mass, and a the Bohr radius. The muons and the atom interact via the Coulomb potential. Neglect recoil of the atom.

(a) [**10 points**] Derive a general expression for the differential cross section of scattering with the excitation of the atom from the ground state $|0\rangle$ to a state $|n\rangle$ in terms of the wave functions, $\psi_0(\vec{r})$ and $\psi_n(\vec{r})$, and the energies, E_0 and E_n, of these states. Here n denotes all quantum numbers of the excited state.

Directions: Taking into account that the muons have high velocity, use the Fermi Golden Rule to calculate the scattering rate. This approximation is analogous to the Born approximation.

Useful formula:

$$\int \frac{e^{-i\vec{q}\cdot\vec{r}}}{|\vec{r}-\vec{R}|}\, d^3\vec{r} = \frac{4\pi e^{-i\vec{q}\cdot\vec{R}}}{q^2} \tag{2}$$

(b) [**8 points**] Using your solution of Part (a), calculate the differential cross-section $d\sigma/d\Omega$ of scattering on the hydrogen atom with its excitation from the 1s state to the 2s state.

Potentially useful:

$$\psi_{1\mathrm{s}} = \frac{e^{-r/a}}{\sqrt{\pi a^3}} \qquad \psi_{2\mathrm{s}} = \frac{e^{-r/2a}}{\sqrt{8\pi a^3}}\left(1 - \frac{r}{2a}\right) \tag{3}$$

(c) [**7 points**] Using your solution of Part (b), calculate the total cross-section $\sigma_t = \int d\Omega (d\sigma/d\Omega)$ of scattering on the hydrogen atom with its excitation from the 1s state to the 2s state. When calculating the integral, make a reasonable approximation taking into account conditions (1).

Problem II.4

The wave-function for a spin-1/2 particle is written as a two-component spinor

$$\Psi(x) = \begin{pmatrix} \psi_\uparrow(x) \\ \psi_\downarrow(x) \end{pmatrix}.$$

The time-reversal operator for this system is written as $\Theta = i\sigma_y K$ where K is the complex conjugation operator. Consider the single-particle Hamiltonian

$$H = ap^4 + bp^3\sigma_x + cp\sigma_z - de^{-x^2},$$

where x is the position operator, p is the momentum operator, and $a, b, c, d \geq 0$ are constants.

(a) [**8 points**] By considering the action of the operators p, σ_x, σ_z and e^{-x^2} on a wave-function $\Psi(x)$ detemine whether each of these operators are even or odd (i.e. symmetric or anti-symmetric) under time-reversal.

(b) [**2 points**] Use the symmetries of the operators in the last part to show that H is symmetric (even) under time-reversal.

(c) [**5 points**] By considering the action of Θ on the wave-function $\Psi(x)$ show that $\Theta^2 = -1$.

(d) [**5 points**] (Kramer's theorem) Consider an energy eigenstate $\Psi_n(x)$ with energy eigenvalue E_n. Show that $\Psi'_n(x) = \Theta\Psi_n(x)$ is an eigenstate with the same energy eigenvalue, which is orthogonal to $\Psi_n(x)$.

(e) [**5 points**] When $a = 0$, the eigenvalue spectrum has no bound states. For $|x| \gg 1$ (so that the potential term $e^{-x^2/2}$ can be neglected), its scattering states are essentially those of a free particle

$$\Psi_p(x) = e^{ipx/\hbar} \begin{pmatrix} \psi_\uparrow \\ \psi_\downarrow \end{pmatrix}.$$

It is paired by Kramers' theorem with $\Psi'_{-p}(x) = \Theta\Psi_p(x)$, which is energetically degenerate with $\Psi_p(x)$ but moving in the opposite direction. Calculate the matrix element $\langle \Psi'_p | e^{-x^2/2} | \Psi_p \rangle$ to show that the back-scattering rate vanishes according to Fermi's Golden rule.

Problem II.5

Consider a system of $\mathcal{N}_d$ electrons/m^3, each of which can occupy either a bound-state level with energy $-\epsilon_d$ (called a "neutral donor") or a free-particle continuum state with energy $\hbar^2 k^2/2m$ (leaving behind an "ionized donor"), where m is the electron mass.

(a) [**5 points**] What are the occupancies $\mathcal{N}_d^0/\mathcal{N}_d$ of neutral donors (singly occupied bound states) and $\mathcal{N}_d^+/\mathcal{N}_d$ of ionized donors (vacant bound states), respectively, in terms of reciprocal thermal energy $\beta = 1/k_B T$ and chemical potential μ? Be sure to take degeneracy into account: there are two ways to get a neutral donor (spin up or down) but only one way to have a vacant, positively charged donor.

(b) [**2 points**] Show that to get sensible behavior as $T \to 0$ the chemical potential must lie above $-\epsilon_d$.

(c) [**4+1 points**] Now we consider the dispersion of continuum states. Show that the density of states (per volume) $\mathcal{G}(\epsilon)$ of the free-electron gas of the continuum can be written $\mathcal{G}(\epsilon) = A\epsilon^{1/2}$, justifying the exponent 1/2. From dimensional arguments, what are the units of A? For full credit (just one point, not worth the time if you do not know), find the numerical factors in A.

(d) [**3+3 points**] i) Write an expression for the density of electrons in the continuum, $\mathcal{N}_c$. ii) What is the low-temperature limit of $\mathcal{N}_c$, assuming $\mu < 0$. Note that $\Gamma(\frac{3}{2}) = \int_0^\infty x^{1/2} \exp(-x) dx = \sqrt{\pi}/2$.

(e) [**3+4 points**] i) Give the expression that determines the value of chemical potential μ. ii) In the low-temperature limit, find an explicit solution for μ.