UNIVERSITY OF MARYLAND
Department of Physics
College Park, Maryland

PHYSICS Ph.D. QUALIFYING EXAMINATION
PART II

January 20, 2017 9:00 a.m. – 1:00 p.m.

Do any four problems. Each problem is worth 25 points.
Start each problem on a new sheet of paper (because different faculty members will be grading each problem in parallel).

Be sure to write your Qualifier ID ("control number") at the top of each sheet — not your name! — and turn in solutions to four problems only. (If five solutions are turned in, we will only grade # 1 - # 4.)

At the end of the exam, when you are turning in your papers, please fill in a "no answer" placeholder form for the problem that you skipped, so that the grader for that problem will have something from every student.

You may keep this packet with the questions after the exam.

Problem II.1

Consider a particle of mass m that is subject to a 1-dimensional potential along the x-axis described by:

$$V(x) = -\alpha\delta(x),\ \alpha > 0$$

(a) **[2 points]** What are the units of α? Write down the time-independent Schrödinger equation for this situation. What can you say about the particle behavior for the condition that the energy $E > 0$? For $E < 0$?

(b) **[5 points]** Consider the condition $E < 0$ for some as-yet unknown E. Compute a normalized wavefunction in terms of this given E.

(c) **[4 points]** Use an appropriate boundary condition at $x = 0$ to determine E in terms of α. State how the value of α affects the shape of the wave function; in particular, consider the effect of small α vs. large α.

(d) **[6 points]** Now consider the particle in the one-dimensional potential

$$V(x) = -\frac{\alpha}{2}[\delta(x - \frac{a}{2}) + \delta(x + \frac{a}{2})].$$

Compute the ground state energy and wavefunction for this potential. You may leave the energy eigenvalue condition as a transcendental equation, and you do not have to normalize the wavefunction.

(e) **[4 points]** Suppose $a \to 0$. Find the ground state energy in this limit. How does this energy compare to that in part (c), and why?

(f) **[4 points]** Suppose $a \to \infty$. Find the ground state energy in this limit. Sketch the wavefunction in this limit.

Problem II.2

Consider a particle of mass m and charge q, confined inside a 2-dimensional *square* region of side L (i.e., with infinite potential outside). The aim of this problem is to find the energy levels of the particle in the presence of a constant electric field $\mathbf{E}$ oriented in the plane.

(a) **[5 points]** Begin with the unperturbed case, in the absence of the electric field. What are the normalized wavefunctions and energies of the particle?

(b) **[6 points]** Use perturbation theory to compute the shift of the ground state energy due to the electric field $\mathbf{E} = E_x \mathbf{x} + E_y \mathbf{y}$, where $\mathbf{x}$ and $\mathbf{y}$ are unit vectors in the x- and y-directions, respectively.

(c) **[5 points]** Now consider the (degenerate) first excited state in the unperturbed spectrum. Can the perturbation cause a mixing between these degenerate levels?

(d) **[6 points]** Compute the shift(s) due to the electric field in the energy of the first excited state(s).

(e) **[3 point]** Compute the energy shift(s) due to the electric field for an *arbitrary* energy level, including those that are degenerate.

Problem II.3

In three dimensions, a spin-1/2 particle with mass m moves in the x-direction in a potential given by:

$$V(x) = V_0 \sigma_z \text{ for } x > 0 \text{ and } V(x) = 0 \text{ for } x \leq 0.$$

Take the σ matrices to be:

$$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}; \quad \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}; \quad \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

(a) [**6 points**] Suppose a beam of particles comes from $-\infty$ with velocity in the positive x direction with energy $E > V_0 > 0$ and is in an eigenstate of σ_x with eigenvalue $+1$. Write down the general eigenstate of such an incoming beam.

(b) [**4 points**] Write down the general solution for the transmitted and the reflected beams.

(c) [**8 points**] What are the boundary conditions at $x = 0$? Write down the equations that must be satisfied by the amplitudes appearing in part (a) and (b).

(d) [**7 points**] Under what condition is the reflected wave an eigenstate of σ_x? Under what condition is the transmitted wave is an eigenstate of σ_x? Explain your answer.

Problem II.4

Consider a system in which particles of mass M are confined to the surface of a sphere of radius a. The dynamics of any **one** particle is described by the Hamiltonian

$$\mathcal{H}_i = \frac{L_i^2}{2Ma^2}, \tag{1}$$

where L_i represents the angular momentum operator for the ith particle.

(a) [**4 points**] Write down the orbital angular momentum quantum numbers ℓ, m of the single-particle ground state and first excited states. Also write down the energies of these states.

(b) [**4 points**] Now assume the system consists of **two identical, non-interacting** spin-1/2 fermions A and B. Write down the two-particle wave function that describes the ground state, including both the spin and orbital degrees of freedom, and respects indistinguishability.

(c) [**4 points**] For this state, write down the total energy E, total orbital angular momentum L, total spin angular momentum S and total angular momentum J.

(d) [**6 points**] For the same two-fermion system, write down a set of two-particle wave functions that describe all the first excited states.

(e) [**7 points**] Transform to a basis in which the wave functions are also eigenstates of the total angular momentum operator J. In this basis, for each wave function, write down the total energy E, total orbital angular momentum L, total spin angular momentum S and total angular momentum J.

Possibly useful:

$$J_-|J, J_z\rangle = \sqrt{(J + J_z)(J - J_z + 1)}|J, J_z - 1\rangle$$

Problem II.5

A Bose-Einstein condensate (BEC) occurs when, under suitable conditions, a macroscopically large number of bosons populate the ground state. Consider a system of N identical non-interacting non-relativistic spinless bosons of mass m in an isolated space with rigid walls at a temperature T (for three-dimensional confinement, volume V; for two-dimensional confinement, area A; for one-dimensional confinement, length L).

(a) [**5 points**] Write down an integral expression in terms of E, T and N, used to fix the chemical potential for the three cases of 3D, 2D and 1D. Assume N, V, A and L are very large.

(b) [**5 points**] For 3D confinement, use an appropriate approximation to find an analytic expression for the chemical potential $\mu(T)$ in the classical limit (i.e. high temperature). Show how this quantity varies when T decreases. Write down the condition for occurrence of the BEC. Hint: $\int_0^\infty x^{1/2}e^{-x}dx = \sqrt{\pi}/2$.

(c) [**5 points**] Determine the BEC transition temperature, T_C for the 3D gas.

(d) [**3 points**] Argue qualitatively why BEC phenomena has a quantum origin, i.e. the wave-like nature of the bosons is important for condensation to occur.

(e) [**3 points**] Can BEC occur in the 2D and 1D gases?

(f) [**4 points**] Now consider *massless* photons in equilibrium with a 3D cavity of volume V at temperature T. What is the chemical potential for this gas of photons? Calculate the average total energy, and explain briefly why it is challenging to experimentally realize BEC in a photon gas.

Useful integral:

$$\int_0^\infty \frac{x^{\alpha-1}}{e^x - 1}dx = \Gamma(\alpha)\zeta(\alpha),$$

where $\Gamma(\alpha)$ and $\zeta(\alpha)$ are Gamma function and Riemann zeta function, respectively.
$\Gamma(1)\zeta(1) = \infty$ $\qquad \Gamma(3/2)\zeta(3/2) \approx 1.306\sqrt{\pi}$ $\qquad \Gamma(4)\zeta(4) = \pi^4/15$