UNIVERSITY OF MARYLAND
Department of Physics
College Park, Maryland

PHYSICS Ph.D. QUALIFYING EXAMINATION
PART I

January 22, 2018 9:00 a.m. – 1:00 p.m.

Do any four problems. Each problem is worth 25 points.
Start each problem on a new sheet of paper (because different faculty members will be grading each problem in parallel).

Be sure to write your Qualifier ID ("control number") at the top of each sheet — not your name! — and turn in solutions to four problems only. (If five solutions are turned in, we will only grade # 1 - # 4.)

At the end of the exam, when you are turning in your papers, please fill in a "no answer" placeholder form for the problem that you skipped, so that the grader for that problem will have something from every student.

You may keep this packet with the questions after the exam.

Problem I.1

A mass m moves with initial velocity v_o towards a star of mass M and radius R. Assume the star is at rest, and that the mass starts far away with an impact parameter b. (The impact parameter is how close the mass would come to the center of the star if its path did not deviate from a straight line.) Let the system be governed by Newton's Equations $m\, d\mathbf{v}/dt = \mathbf{F}$ and $d\mathbf{r}/dt = \mathbf{v}$, where the field $\mathbf{F}$ is, as yet, unspecified.

(a) [**3 points**] The vector angular momentum of the mass is $\mathbf{L} = m\mathbf{r} \times \mathbf{v}$, where $\mathbf{r}$ is the position vector from the star center of mass. Calculate $d\mathbf{L}/dt$ and so find the condition on $\mathbf{F}$ under which the angular momentum will be conserved.

(b) [**2 points**] Assume the force field allows conservation of $\mathbf{L}$. Show that the motion is confined to a plane perpendicular to $\mathbf{L}$.

(c) [**4 points**] Assume now that $\mathbf{F}$ is the gravitational force. Using polar coordinates for the mass location, $r(t)$ and $\phi(t)$, write down a pair of equations expressing conservation of energy, E, and angular momentum, L. Substitute so that your equation for E involves L.

(d) [**10 points**] Use your two constants of the motion in (c) to write down an "equivalent 1D problem", i.e. an "equation of motion" for the variable $r(t)$ that does not involve ϕ. Make sure to insert the given initial conditions, in terms of b and v_0. With this information, find an inequality that b must satisfy in order that the mass m should not hit the star of radius R. To keep the algebra manageable, define the constant a as $a = GM/v_0^2$, and note that a has dimensions of length.

(e) [**3 points**] For no forces, the cross-section σ that the star presents to the mass is πR^2. From your results in (d), find the cross-section for striking the star if $a \ll R$ and compare that to σ. Do likewise for when $R \ll a$. To get at the physical meaning, note that the dimensionless parameter a/R is equal (aside from a factor of 2) to the ratio of two relevant energies. What are they, in words?

(f) [**3 points**] Why does this problem matter for the formation of solar systems (short paragraph answer).

Possibly useful: $d(\mathbf{A} \times \mathbf{B}) = d\mathbf{A} \times \mathbf{B} + \mathbf{A} \times d\mathbf{B}$

Problem I.2

A conducting sphere of radius a and surface charge Q spins about the z-axis at angular frequency ω. The charge is fixed on the surface, *i.e.* not flowing relative to the atoms of the sphere.

(a) **[3 points]** What is the surface charge density, σ, in terms of Q? What is the surface current density, expressed in spherical coordinates (with θ representing the angle from the z axis)?

(b) **[4 points]** Write down the magnetostatic equations that govern the magnetic field. Conventionally, we may express $\mathbf{B}$ as the curl of a vector potential. However, state why it is allowable in this case, away from the surface, to write $\mathbf{B}$ as the gradient of a scalar magnetic potential, $\mathbf{B} = \nabla\Phi$. Finally, obtain the partial differential equation satisfied by Φ away from the surface.

(c) **[6 points]** Because of the surface current at $r = a$, the radial and tangential ($\hat{\boldsymbol{\theta}}$) components of $\mathbf{B}$, inside and outside the sphere, satisfy boundary conditions across the surface charge layer. Using the magnetostatic equations, obtain these conditions. Then express these in terms of partial derivatives of Φ, in spherical coordinates.

(d) **[8 points]** Solve for Φ, apply the boundary conditions across the surface, and find Φ everywhere in terms of a, σ, and ω. Also, obtain $\mathbf{B}$. Make a sketch of $\mathbf{B}$ lines inside and outside the sphere.

(e) **[4 points]** Suppose the sphere is charged up to voltage V. What is the value of $\mathbf{B}$ at the center if the radius is 0.1 m, voltage is 10 kV, and the spin rate is 10^4 rpm.

Potentially useful:

In spherical coordinates, $\boldsymbol{\nabla} = \frac{\partial}{\partial r}\hat{\boldsymbol{r}} + \frac{1}{r}\frac{\partial}{\partial\theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\hat{\boldsymbol{\phi}}$

$P_0(x) = 1, \quad P_1(x) = x, \quad P_2(x) = \frac{1}{2}(3x^2 - 1)$

$\epsilon_0 = 8.85 \times 10^{-12}\ \frac{C^2}{N.m^2}$

$\mu_0 = 4\pi \times 10^{-7}\ \frac{N}{A^2}$

$c = \frac{1}{\sqrt{\mu_0\epsilon_0}} = 3 \times 10^8\ \frac{m}{s}$

Problem I.3

The *Otto cycle* is an idealization of the processes in an internal combustion engine with a variable-volume chamber controlled by the motion of a piston head inside a cylinder. The cycle consists of four "strokes", in which the piston moves between two extrema, V_{min} and V_{max}. Beginning with the chamber at minimum volume, they are: 1. an isobaric (constant pressure) intake stroke filling the chamber with air and fuel; 2. an isentropic (constant entropy or 'adiabatic') compression stroke, then isochoric (constant volume) ignition; 3. isentropic expansion (power) stroke, and finally 4. isochoric escape through the exhaust port, then isobaric exhaust stroke.

(a) [**4 points**] Assuming the fuel/air mixture and its reaction products behave as diatomic ideal gases, what is the exponent γ in the relationship between pressure and volume $p \propto V^{-\gamma}$ during the isentropic compression and expansion stages?

(b) [**6 points**] Draw the path through pressure-volume $p - V$ space, labeling all *six* segments and indicating direction. Note that the isobaric intake and exhaust strokes lie along the same path in this idealization.

(c) [**3 points**] As noted above, the isobaric intake and exhaust strokes lie along the same path in this idealization. How do those paths differ in a real engine, and what is the consequence? Why would you expect an increase in engine performance if you enlarge the diameter of an exhaust pipe, or remove the intake air filter?

(d) [**5 points**] Plot the isochoric and isentropic paths through entropy-temperature $S - T$ space. Graphically explain why the Otto cycle can't reach the Carnot efficiency.

(e) [**3 points**] Heat flows into and out of the chamber during the isochoric paths. Assuming a constant heat capacity, write the efficiency η_{Otto} in terms of the four temperatures at the "corners" of the main cycle. Use T_a for the temperature at the start of the isentropic compression segment, T_b at the start of the ignition segment, T_c at the start of the isentropic expansion, and T_d for the temperature at the start of the exhaust escape segment.

(f) [**4 points**] Use the quantity conserved along the isentropic paths to relate temperature to volume, and hence express the efficiency as a function of compression ratio $r = V_{\text{max}}/V_{\text{min}}$. Calculate or estimate the thermodynamic efficiency of a modern engine with $r \sim 10$.

Problem I.4

Compton's study of the scattering of x-ray photons off of massive particles (electrons) gave support to the picture of photons as massless particles carrying energy and momentum. This was seen via the Compton effect: the increase in the wavelength of scattered radiation according to the Compton formula $\lambda_f - \lambda_i = \lambda_{\text{Compton}}(1 - \cos\theta^{\text{lab}})$, where θ^{lab} is the scattering angle of the photon in the lab frame, $\lambda_{\text{Compton}} = \frac{h}{mc}$, and m is the mass of the target particle.

(a) **[5 points]** The Compton formula applies to all photons including optical photons; diffraction gratings to measure the wavelength of light were readily available throughout much the 19th century. Yet, the Compton effect was not discovered experimentally until the 1920s, after the development of diffractive techniques using crystals to measure the wavelength of x-rays. Why did it only became practical experimentally to detect the Compton effect after measurements using x-rays become available?

In the remainder of the problem the scattering of a photon off of a massive particle will be described using the energy momentum 4-vectors in both the lab and center of mass frames and then the Compton formula derived. The notation used here is the 0th component of the vector is energy and units with $c = 1$ will be used until part (f). In the lab frame the incident photon has energy, E_γ^{lab} and is moving in the positive x-direction, the charged particle has mass, m and is initially at rest and for simplicity assume that the scattering is in the x-y plane.

(b) **[5 points]** As a first step consider the center-of-mass frame where the total momentum is zero both before and (due to momentum conservation) after the scattering. Show that energy will be conserved if the magnitude of the photon's incident momentum is equal to the magnitude of its final momentum (with the energy of the photon and the massive particle being given by the relativistic energy momentum relation).

Part (b) means that the initial and final energy-momentum 4-vectors for the photon in the c.m. frame can be expressed: $p_\gamma^{\text{incident c.m.}} = \begin{pmatrix} E_\gamma^{\text{c.m.}} \\ E_\gamma^{\text{c.m.}} \\ 0 \\ 0 \end{pmatrix}$, $p_\gamma^{\text{final c.m.}} = \begin{pmatrix} E_\gamma^{\text{cm}} \\ E_\gamma^{\text{c.m.}} \cos\theta^{\text{c.m.}} \\ E_\gamma^{\text{c.m.}} \sin\theta^{\text{c.m.}} \\ 0 \end{pmatrix}$; where $\theta^{\text{c.m.}}$ is the scattering angle in the c.m. frame.

(c) **[5 points]** Find the 4-by-4 matrix for the Lorentz transformation from the center-of-mass frame to the Lab frame. Express this in terms of $E_\gamma^{\text{c.m.}}$ and m.

Problem continues on next page, including possibly-useful notes

(d) [**3 points**] Apply the Lorentz transformation of part (c) to $p^{\text{incident c.m.}}$ and $p^{\text{incident c.m.}}$ to obtain $p_\gamma^{\text{incident lab}} = \begin{pmatrix} E_\gamma^{\text{lab incident}} \\ E_\gamma^{\text{lab incident}} \\ 0 \\ 0 \end{pmatrix}$, $p_\gamma^{\text{final lab}} = \begin{pmatrix} E_\gamma^{\text{lab final}} \\ E_\gamma^{\text{lab final}} \cos(\theta^{\text{lab}}) \\ E_\gamma^{\text{lab final}} \sin(\theta^{\text{lab}}) \\ 0 \end{pmatrix}$ where θ^{lab} is the photon scattering angle in the lab frame. Express your answer in terms of m, $E_\gamma^{\text{c.m.}}$ and $\theta^{\text{c.m.}}$

(e) [**3 points**] Show that $E_\gamma^{\text{lab incident}} - E_\gamma^{\text{lab final}} = \frac{E_\gamma^{\text{lab incident}} E_\gamma^{\text{lab final}}}{m} \left(1 - \cos\theta^{\text{lab}}\right)$.

(f) [**4 points**] Derive the Compton formula, inserting appropriate factors of c and Planck's constant.

Possibly useful:

Compton wavelength of proton$=1.32 \times 10^{-15}$ m
Compton wavelength of electron$=2.43 \times 10^{-12}$ m
Wavelength of visible light $\sim 3.9 \times 10^{-7} - 7.0 \times 10^{-7}$ m
Wavelength of x-rays $\sim 10^{-11} - 10^{-13}$ m

Problem I.5

Two parallel semi-transparent mirrors, separated a distance d, enclose a non-dispersive material of refraction index n. Consider electromagnetic waves of radial frequency ω propagating at normal incidence to the mirror plane.

(a) [**2 points**] What is the optical phase θ accumulated during propagation from one mirror to the other, in terms of d, n, ω, and the speed of light c?

(b) [**2 points**] Each mirror has an electric-field transmission amplitude t and a reflection amplitude r. Assuming the mirrors are lossless, use conservation of energy to relate these two quantities.

(c) [**5 points**] A continuous beam of coherent light which passes through the first mirror will bounce back and forth between the mirrors, interfering with new incoming light each time it bounces on the first mirror. This arrangement is called a Fabry-Perot cavity. Considering the effects of interference, show that the power transmission coefficient (ratio of transmitted to incident electromagnetic intensity) is $T = t^4/(1 + r^4 - 2r^2 \cos 2\theta)$.

(d) [**3 points**] Assume that the mirrors are highly reflective so that $r \to 1$. What is the transmission coefficient when $\omega = \omega_0 = \frac{\pi c}{nd}$?

(e) [**3 points**] Use an appropriate approximation to find the functional dependence of $T(\omega)$ in the vicinity of ω_0. You may express it in terms of $\delta\omega \equiv \omega - \omega_0$.

(f) [**4 points**] Again assuming nearly perfect reflection, determine the "quality factor" for this resonance, Q, in terms of r.

(g) [**3 points**] Now assume that the media inside the Fabry-Perot is slightly non-linear, such that the resonance frequency depends on unitless intensity α as $\omega_0 \to \omega_0(1 - \alpha T)$. Sketch T as a function of ω for $\alpha \ll 1/Q$ and $\alpha \gtrsim 1/Q$.

(h) [**3 points**] Plot T at a fixed $\alpha \approx 1/Q$, as a function of frequency ω, swept through ω_0. Describe the difference between increasing and decreasing ω.