UNIVERSITY OF MARYLAND
Department of Physics
College Park, Maryland

PHYSICS Ph.D. QUALIFYING EXAMINATION
PART II

January 23, 2018 9:00 a.m. – 1:00 p.m.

Do any four problems. Each problem is worth 25 points.
Start each problem on a new sheet of paper (because different faculty members will be grading each problem in parallel).

Be sure to write your Qualifier ID ("control number") at the top of each sheet — not your name! — and turn in solutions to four problems only. (If five solutions are turned in, we will only grade # 1 - # 4.)

At the end of the exam, when you are turning in your papers, please fill in a "no answer" placeholder form for the problem that you skipped, so that the grader for that problem will have something from every student.

You may keep this packet with the questions after the exam.

Problem II.1

Consider a particle of mass m and charge q confined to move on a one-dimensional ring of radius R. Model the quantum state of the particle by a wavefunction $\psi(x)$ subject to periodic boundary conditions, $\psi(x+2\pi R) = \psi(x)$. The ring is threaded by a solenoid of radius $L < R$ generating a magnetic flux Φ. The Hamiltonian of the particle is

$$H = \frac{(p - qA)^2}{2m}$$

where $A = \frac{\Phi}{2\pi R}$ is the component of the vector potential along the ring.

(a) [**5 points**] For $\Phi = 0$, show that $\psi_k(x) = e^{ikx}$ is a solution of the time-independent Schrodinger equation and compute the energy E as a function of k. What values of k are consistent with the boundary conditions?

(b) [**5 points**] Now suppose $\Phi \neq 0$. Repeat the analysis from part (a) to find the energy levels as a function of Φ.

(c) [**5 points**] How do the energy levels at $\Phi = 0$ compare to those at $\Phi = \frac{2\pi\hbar}{q}$? How do the energy levels at Φ compare with those at $-\Phi$? If an experimental apparatus can only measure the energy (and not the value of k), what can be determined about the magnetic flux Φ?

Now suppose we place a very high barrier of width W which prevents the particle from moving around the entire ring. Model this barrier as a potential $V(x)$ of the form

$$x \in (0, 2\pi R - W) : V(x) = 0,$$
$$x \in [2\pi R - W, 2\pi R] : V(x) = \infty.$$

(d) [**2 points**] In the presence of the barrier, what conditions does a finite energy state $\psi(x)$ obey at $x = 0$ and $x = 2\pi R - W$? Consider another wavefunction $\tilde{\psi}(x)$ defined by $\psi(x) = e^{i\alpha x}\tilde{\psi}(x)$. What conditions does $\tilde{\psi}(x)$ obey at $x = 0$ and $x = 2\pi R - W$?

(e) [**5 points**] Find the energy levels of the particle in the presence of the barrier. You may find it useful to work in terms of $\tilde{\psi}$ and choose α to simplify the problem.

(f) [**3 points**] Consider the limit of a very thin barrier, so that $0 < W \ll R$. How do the energy levels depend on Φ? If an experimental apparatus can only measure the energy, what can be determined about the magnetic flux Φ? Interpret your result physically.

Problem II.2

(a) [**8 points**] Derive the energies and normalized eigenstates of a particle in an infinite square well which extends from $x = 0$ to $x = a$.

Two identical spinless bosons are placed in the infinite square well. They interact weakly with one another, via the potential

$$V(x_1, x_2) = -a\, V_0\, \delta(x_1 - x_2),$$

where V_0 is a constant with the dimensions of energy, and a is the width of the well.

(b) [**8 points**] Ignoring the interaction between the particles, find the ground state and the first excited state of the two-particle system – both the wave functions and the associated energies.

(c) [**9 points**] Use first-order perturbation theory to estimate the effect of the particle-particle interaction on the energies of the ground state and the first excited state.

Possibly useful:

$\int_0^\pi \sin^2 y\, dy = \pi/2$
$\int_0^\pi \sin^4 y\, dy = 3\pi/8$
$\int_0^\pi \sin^6 y\, dy = 5\pi/16$

Problem II.3

Consider an incident wave packet, $\psi_{\text{in}}(\boldsymbol{r},t)$, with an energy spread ΔE, scattering off a target (e.g., neutrons scattering off the iridium atom), and a resonance of angular momentum $l\hbar$, width Γ_R and energy E_R is measured in an experiment. While the scattering is resonant in partial wave l, the contribution from other partial waves to the scattering amplitude is assumed to be negligible.

(a) [**5 points**] How can a resonance be identified experimentally?

The following steps will guide you to identify another signature of a resonance when the resonance is broad compared with the energy spread of the incident wave packet, $\Gamma_R \gg \Delta E$.

The wavefunction describing the elastic scattering of a wave packet off a target at origin is

$$\psi(\boldsymbol{r},t) = \psi_{\text{in}}(\boldsymbol{r},t) + \psi_{\text{sc}}(\boldsymbol{r},t), \tag{1}$$

where $\psi_{\text{sc}}(\boldsymbol{r},t)$ is given by

$$\psi_{\text{sc}}(\boldsymbol{r},t) = \int \frac{d^3p}{(2\pi)^{3/2}}\phi(p\hat{\boldsymbol{z}})\frac{f(p\hat{\boldsymbol{z}},p\hat{\boldsymbol{r}})}{r}e^{ipr-iEt}. \tag{2}$$

Here, $\phi(\boldsymbol{p})e^{-iEt}$ is the Fourier transform of $\psi_{\text{in}}(\boldsymbol{r},t)$ to three-momentum $\boldsymbol{p}$. For simplicity the momentum of the incident wave packet is assumed to be peaked along the z direction, $\boldsymbol{p} = p\hat{\boldsymbol{z}}$. Additionally, $r = \sqrt{x^2+y^2+z^2}$ and E is the energy of the incident wave. Recall that the scattering amplitude f has a partial-wave expansion of the form

$$f(\boldsymbol{p},\boldsymbol{p}') = \sum_l (2l+1)\frac{e^{i\delta_l}\sin(\delta_l)}{p}P_l(\boldsymbol{p}\cdot\boldsymbol{p}') \tag{3}$$

in terms of phase shifts δ_l, where P_l is the Legendre polynomial of degree l.

(b) [**8 points**] Assuming the conditions stated for the resonance under consideration in this problem, show that at large r:

$$\psi_{\text{sc}}(\boldsymbol{r},t) \to \frac{f(p\hat{\boldsymbol{z}},p\hat{\boldsymbol{r}})}{r}\psi_{\text{in}}((r+\delta_l')\hat{\boldsymbol{z}},t), \tag{4}$$

up to an unimportant overall phase factor. δ_l' denotes the derivative of δ_l with respect to p.

Hint: Assume the wave packet is sharply peaked around some energy $\overline{p}$ in the region close to the resonance momentum, p_R and Taylor expand $e^{i\delta_l(p)}$ and $\sin(\delta_l(p))$ around $\overline{p}$.

(c) [**4 points**] Interpret the result given in Eq. (4) by comparing the scattering wave with the incident wave at i) the same time, ii) the same position. Ignore the overall difference in the amplitude of the waves and be as quantitative as possible.

(d) [**4 points**] Assume that $\delta_l' < 0$. What is the condition on δ_l' such that the causality is not violated? Hint: Keep in mind that the target has a finite size d in the z direction.

(e) [**2 points**] Assume that $\delta_l' > 0$. Is it necessary to ensure that a condition similar to that in part (b) holds?

(f) [**2 points**] A physical interpretation of a resonant scattering is in terms of a metastable state, in which the projectile is captured temporarily by the target before it decays and re-emits the scattering wave. Which of the scenarios discussed above is consistent with this physical picture, part (c) or (d)?

Problem II.4

The parity operator in one dimension, P_1, reverses the sign of the position coordinate, x:

$$P_1\psi(x) = \psi(-x).$$

(a) [**5 points**] Find the eigenvalues and eigenfunctions of the parity operator. Is the parity operator Hermitian? Explain.

(b) [**5 points**] Calculate the commutator of P_1 with the position operators, $[P_1, \hat{x}]$, and with the position operator squared, $[P_1, \hat{x}^2]$.

(c) [**3 points**] For a particle moving in a one-dimensional potential $V(x)$, what condition must $V(x)$ satisfy for parity to be a symmetry (that is, $[H, P_1] = 0$)? Explain.

(d) [**4 points**] Consider $V(x) = k|x|$. Is the lowest energy eigenstate also an eigenfunction of P_1? If so, what is its parity? Explain briefly.

For the following, the angular momentum operator in 3-dimensions is given by $\hat{\mathbf{L}} = \hat{\mathbf{x}} \times \hat{\mathbf{p}}$.

(e) [**3 points**] Calculate the commutator of $\hat{\mathbf{L}}$ with the position operators $\hat{\mathbf{x}}$. That is, calculate $[\hat{L}_i, \hat{x}_j]$, where i, j run over the 3 Cartesian directions x, y and z. (Hint: first consider $\hat{L}_x$, then generalize from that.) Also calculate the commutator of $\hat{\mathbf{L}}$ with $\hat{\mathbf{x}}^2$.

(f) [**5 points**] For a spinless particle moving in a 3-dimensional potential $V(\mathbf{x})$, what condition must the potential satisfy for the angular momentum to be a generator of a symmetry? Explain. If $V(\mathbf{x})$ indeed satisfies this condition, what are the implications for the eigenstates of H?

Problem II.5

A particle of charge e and mass m moves in an external magnetic field along the z-direction with magnitude B, in a volume $V = L^3$ with $L \gg \frac{mc}{eB}$.

(a) [**5 points**] Using the gauge $\boldsymbol{A} = (A_x, 0, 0)$, show that the time-independent Schrödinger equation can be written as

$$\frac{\hbar\omega_c}{2}\left[\left(-i\frac{\partial}{\partial x'} + y'\right)^2 - \frac{\partial^2}{\partial y'^2} - \frac{\partial^2}{\partial z'^2}\right]\Psi = E\Psi.$$

where $\omega_c = \frac{eB}{m}$ and the unitless coordinates $x' = x/\ell$, $y' = y/\ell$, and $z' = z/\ell$. Give the "magnetic length" ℓ in terms of ω_c and other quantities in the problem.

(b) [**5 points**] Use the ansatz $\Psi(\boldsymbol{r}) \sim e^{ik_x\ell x'}e^{ik_z\ell z'}\phi(y')$ to write an equation for $\phi(y')$. What is the energy eigenvalue spectrum?

(c) [**5 points**] Use the finite size of the volume to determine the degeneracy of each state with unique eigenenergy.

(d) [**5 points**] Use these levels to evaluate the single-particle partition function Z at high temperature T in the limit where $\hbar\omega_c \ll k_B T$ (k_B is the Boltzmann constant). Retain the lowest-order term dependent on magnetic field.

(e) [**5 points**] Use the partition function to calculate the magnetic susceptibility χ at high temperature. Show that it is diamagnetic for small fields and obeys Curie's law, $\chi \propto T^{-1}$.

Potentially useful:
$\int_0^\infty e^{-x^2}dx = \sqrt{\pi}/2$
$\ln(1+x) \approx x, \quad x \ll 1.$
$\sinh x \approx x + x^3/6, \quad x \ll 1.$