UNIVERSITY OF MARYLAND
Department of Physics
College Park, Maryland

PHYSICS Ph.D. QUALIFYING EXAMINATION
PART I

January 21, 2016 9:00 a.m. – 1:00 p.m.

Do any four problems. Each problem is worth 25 points.
Start each problem on a new sheet of paper (because different faculty members will be grading each problem in parallel).

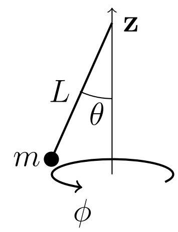
Be sure to write your Qualifier ID ("control number") at the top of each sheet — not your name! — and turn in solutions to four problems only. (If five solutions are turned in, we will only grade # 1 - # 4.)

At the end of the exam, when you are turning in your papers, please fill in a "no answer" placeholder form for the problem that you skipped, so that the grader for that problem will have something from every student.

You may keep this packet with the questions after the exam.

Problem I.1

A spherical pendulum consists of a point mass, m, suspended by a massless string of length L. Let the potential energy of the pendulum be zero at $\theta = 0$.



(a) (i) [**4 points**] Write down the Lagrangian for the system.

(ii) [**5 points**] What quantities are conserved and why are they conserved?

(b) If ϕ is a constant, the pendulum becomes a plane pendulum. Suppose that θ_m is the maximum amplitude for θ in this motion.

(i) **[6 points]** Using your answer from part (a), derive an integral expression for the period of this plane pendulum.

(ii) [**5 points**] Now suppose that θ_m is small, and, by making the appropriate small angle approximations, evaluate the integral to get an explicit expression for the period. Compare your answer to the result obtained from application of Newton's 2^{nd} law to a simple pendulum under the same small-angle assumption.

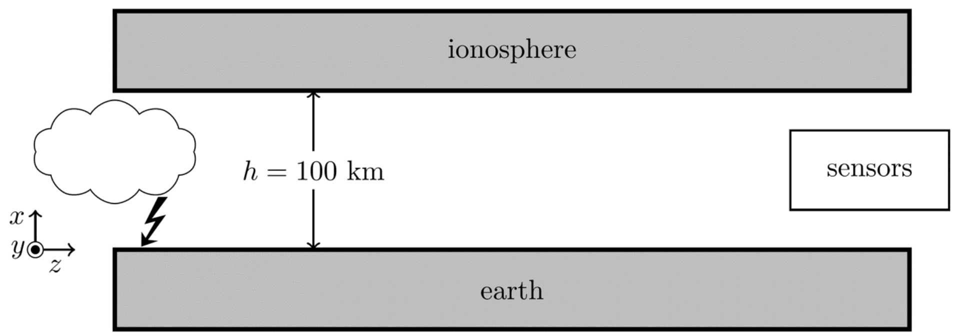
(c) [**5 points**] If θ is a constant, the pendulum becomes a conical pendulum. This means that the mass, m, moves on a circular trajectory. Use your answer from part (a) to find $d\phi/dt$ in terms of θ for this case.

Possibly useful information: $\int_0^1 \frac{dx}{\sqrt{1-x^2}} = \pi/2$.

Problem I.2

A lightning discharge generates a broad spectrum of electromagnetic (EM) waves at frequencies of a few Hz to tens of kHz. The emitted EM waves propagate to the far-field, guided by the conducting earth (mostly seawater) and the ionosphere above the earth at height $h \approx 100$ km (see figure below).

We can model this scenario as a parallel-plate waveguide with infinitely large width $W (W \gg h)$ in the y direction so that any fringe effects can be ignored ($\partial/\partial y = 0$). We also assume that the plates are perfect conductors ($\sigma = \infty$) and the medium in-between is vacuum. We use this model to explore the properties of the EM waves propagating along the z-direction and then detected by sensors in the far-field, many wavelengths away from the discharge.



(a) [**5 points**] Starting from the wave equation, derive the dispersion relation (k_z versus ω) for the TE modes (i.e., $\mathbf{E} = (0,\, E_y,\, 0)$) of angular frequency ω. Is there any cut-off frequency, f_c, below which the TE mode does not propagate? If so, find f_c.

(b) [**5 points**] Find the dispersion relation (k_z versus ω) for the TM modes (i.e., $\mathbf{B} = (0,\, B_y,\, 0)$) of angular frequency ω. Is there any cut-off frequency f_c, below which the TM mode does not propagate? If so, find f_c.

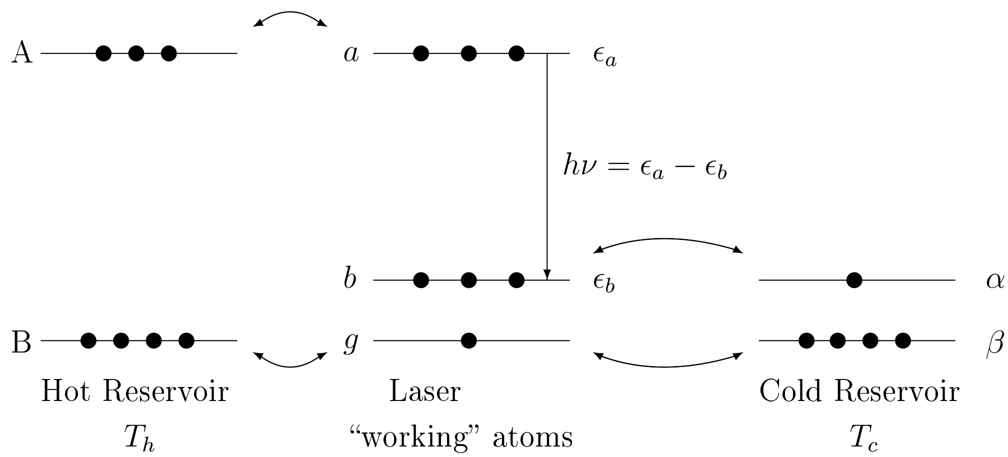
(c) [**5 points**] Find the phase velocity (v_p) and group velocity (v_g) of an EM wave propagating in the z direction for both TE and TM modes. Express them as a function of angular frequency ω.

(d) [**5 points**] Assume EM waves at frequencies of $f = 100$ Hz and 2 kHz are received by sensors located 3000 km away from a lightning strike. For each frequency, determine its arrival time(s) and polarization mode(s) (TE, TM, or both). Assume that all frequencies were radiated simultaneously and propagated at the group velocity.

(e) [**5 points**] In reality, the conductivities of sea water and ionosphere are finite ($\sigma_{\text{seawater}} \approx 4(\Omega\text{m})^{-1}$ and $\sigma_{\text{ionosphere}} \approx 10^{-4}(\Omega\text{m})^{-1}$). Argue how good our perfect conductor assumption is for the frequency at 100 Hz. For simplicity, use $\varepsilon \approx 8.8 \times 10^{-12}$ F/m for both sea water and ionosphere.

Problem I.3

We can think of a laser functioning as a quantum heat engine, with a "hot" reservoir of atoms at a temperature T_h that pump the "working" atoms from the ground state g to an upper level a and a "cold" reservoir at temperature T_c that transfers atoms from the lower lasing level b to the ground state. We can schematically depict the situation as in the figure below:



(a) [**6 points**] The number of atoms in levels a and b obey rate equations (equations for $\dot{n}_a$ and $\dot{n}_b$). We use A and B to denote the hot bath atoms and α and β for the cold bath atoms. Show from simple rate equations that in equilibrium, we have

$$\frac{n_a}{n_g} = \frac{N_A}{N_B} \quad \text{and} \quad \frac{n_b}{n_g} = \frac{N_\alpha}{N_\beta},$$

where the n_j represent the populations in the jth state of the laser and the N_k represent the populations in the kth state of the reservoirs.

(b) [**12 points**] The laser threshold is defined by $n_a = n_b$; i.e., the populations in the upper and lower levels are equal. Use Boltzmann's distribution to show that

$$1 \leq \frac{n_a}{n_b} = \exp\left[-\frac{\epsilon_a}{kT_h} + \frac{\epsilon_b}{kT_c} \right]$$

(c) [**7 points**] For every photon of energy $\epsilon_a - \epsilon_b$ emitted, we must absorb one quantum of energy ϵ_a from a hot bath atom. Thus show that the efficiency of the laser $e_{laser} = 1 - \frac{\epsilon_b}{\epsilon_a}$ is always less than or equal to the Carnot efficiency, $1 - \frac{T_c}{T_h}$.

Problem I.4

If dark matter particles are produced in collisions of ordinary matter, there might be missing energy and/or momentum in the detected collision products. One way to search for this is to examine electron-positron (e^-e^+) collisions in which only a single photon (γ) is detected.

(a) **[5 points]** Show that the process $e^- + e^+ \to \gamma$ cannot occur.

(b) **[7 points]** The process $e^- + e^+ \to \gamma + Z$ can occur, followed by the decay $Z \to \nu + \bar{\nu}$ of the Z-boson to a neutrino-anti-neutrino pair. Like dark matter particles, the neutrinos will not be detected, so this process must be ruled out in a search for dark matter. What is the energy of the emitted photon? Assume the laboratory is the center of mass frame for the collision, and express your answer in terms of the total energy E_{CM} in that frame and the Z-mass m_Z. You may adopt units with $c = 1$.

(c) In the previous process the Z-boson is created as a free particle, which then decays. Instead, the photon and neutrino pair may be produced directly, as $e^- + e^+ \to \gamma + \nu + \bar{\nu}$. (In that case, one can show that the maximum energy of the emitted photon is $E_{CM}/2$, neglecting the very small neutrino mass.) But suppose now that a photon and a pair of dark matter particles, each with mass m_D, is produced directly, $e^- + e^+ \to \gamma + D + \bar{D}$.

(i) **[6 points]** Show that the maximum photon energy for a given E_{CM} occurs when the two dark matter momenta are equal.

(ii) **[4 points]** What is the maximum photon energy? (You may assume the dark matter momenta are equal even if you have not shown it in the previous part.)

(d) **[3 points]** Another potential way to produce dark matter pairs is in quark-antiquark ($q\bar{q}$) annihilation in colliding protons at the LHC. (Instead of looking for a single photon, such experiments can look for a "monojet" originating from emission of a single gluon.) Since the quarks are moving inside the protons, E_{CM} for the quark pair is not fixed by the proton energy. Assume the colliding protons each have energy E_p, and assume the maximum γ-factor for the quarks inside the protons is γ_q in the rest frame of the proton. Find the maximum E_{CM} for the $q\bar{q}$ pair as a function of E_p, γ_q, the quark mass m_q, and the proton mass m_p, assuming $\gamma_q \gg 1$ and $E_p \gg m_p$.

Problem I.5

Consider the one-dimensional wave equation,

$$\frac{\partial^2 f}{\partial t^2} - u^2 \frac{\partial^2 f}{\partial x^2} = 0.$$

(a) [**2 points**] What is the physical meaning of u?

(b) [**2 points**] Derive the relationship (dispersion relation) between angular frequency ω and wave vector k.

(c) [**2 points**] Write down the general solution $f(x, t)$ of the above equation, and state the physical meaning of its parts.

(d) [**1 point**] Suppose initially the wave has a Gaussian form, $f(x, 0) = ae^{-x^2/2\lambda}$. Find the solution at time t, given that the wave packet is traveling along the *positive* x axis.

(e) [**6 points**] Consider a source moving with speed v and emitting a monochromatic wave in a medium where the wave speed is u. Derive the relation between the frequency of the source and that seen by a receiver at rest in the medium and in front of the source. In which direction does the frequency shift (red or blue)?

(f) [**6 points**] Suppose the moving source in (c) emits a Gaussian wave as in (d). How does this Gaussian change when it is observed by the receiver at rest?

(g) Now consider wave propagation in a 3-dimensional medium. When the speed of the source exceeds the speed of the wave, all the emitted waves form a cone of half angle α from the direction of the source motion.

   (i) [**4 points**] Derive the relationship between the angle α and the speeds of the wave and of the source.

   (ii) [**2 points**] Give an example of this in a physical application.