UNIVERSITY OF MARYLAND
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Ph.D. PHYSICS QUALIFYING EXAMINATION - PART II

January 22, 2010 **9 a.m. - 1 p.m.**

Do any four problems. Each problem is worth 25 points.

Put all answers on your answer sheets.

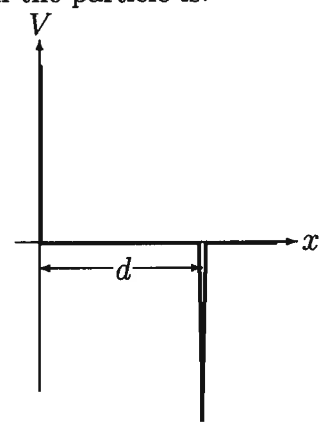
Be sure your Qualifier ID Number is at the top right corner of each sheet and turn in solutions to four problems only. If five solutions are turned in we will grade # 1 - # 4.

Problem II.1

Consider a particle of mass m moving on a straight half-line and under the influence of an attractive delta-function potential. The potential acting on the particle is:

$$V(x) = \begin{cases} \infty & \text{for } x < 0, \\ -\lambda\,\delta(x-d) & \text{for } x > 0, \end{cases}$$

with $\lambda > 0$ and $d > 0$.



(a) Find a relation satisfied by the strength λ and position d of the δ-function potential so that there is, at least, one bound state. **[10 points]**

(b) An observer at $x = \infty$ can infer the existence of the potential by bouncing back waves from the wall at $x = 0$. In the absence of the δ-function potential, the wave function with the energy $E = \hbar^2 k^2/2m$ would be $\psi(x) = \sin(kx)$. However, when the δ-function potential is present, the wave function would acquire a phase shift φ, so that $\psi(x) = \sin(kx + \varphi)$ at large x. Find an equation determining the phase shift φ. **[10 points]**

(c) The transcendental equation determining the phase shift cannot be solved analytically. Assuming that the phase shift is proportional to k at small energies, $\varphi \approx ak$, find the value of a. **[5 points]**

Problem II.2

This problem concerns the hydrogen atom. In the following, ignore spin. The electric dipole moment $\boldsymbol{d}$ between the electron and the proton is the expectation value $\boldsymbol{d} = e\langle \boldsymbol{r} \rangle$, where $\boldsymbol{r}$ is the electron coordinate relative to the proton, and e is the electron charge (negative).

(a) What is $\boldsymbol{d}$ in the ground state of the hydrogen atom? **[3 points]**

(b) Suppose a weak uniform constant electric field $\boldsymbol{F}$ is applied to the hydrogen atom. The Hamiltonian of interaction between the electron and the applied electric field is

$$\hat{H}_F = -e\, \boldsymbol{r} \cdot \boldsymbol{F}. \tag{1}$$

Using perturbation theory, calculate the energy shift δE of the ground state due to the perturbation (1) to the lowest order in $\boldsymbol{F}$. Take into account only the matrix elements of $\hat{H}_F$ between the ground state and the first excited energy level of the hydrogen atom. **[7 points]**

(c) In the presence of the electric field, the dipole moment can be written as

$$\boldsymbol{d} = -\left\langle \frac{\partial \hat{H}_F}{\partial \boldsymbol{F}} \right\rangle = -\frac{\partial\,(\delta E)}{\partial \boldsymbol{F}}, \tag{2}$$

where the second equality follows from the Feynman-Hellmann theorem.

Using Eq. (2) and your result for δE, calculate the dipole moment $\boldsymbol{d}$ of the ground state of the hydrogen atom in the presence of $\boldsymbol{F}$. Show that $\boldsymbol{d} = \alpha \boldsymbol{F}$ and determine the coefficient of proportionality α (which is called the polarizability). **[4 points]**

(d) Without $\boldsymbol{F}$, the first excited energy level of the hydrogen atom is 4-fold degenerate.

Calculate the energy splitting of this level in the presence of the weak electric field to the lowest order in $\boldsymbol{F}$. Comment on what happens to the degeneracy. Using Eq. (2), calculate the values of $\boldsymbol{d}$ in the first excited energy level. **[7 points]**

(e) What is the condition on the strength of $\boldsymbol{F}$ for perturbation theory in $\boldsymbol{F}$ to be valid? Comment both on Parts (b) and (d), for the ground state and the first excited state. **[4 points]**

The next page has some potentially useful information.

$$\int_0^\infty x^n e^{-x} dx = n! \qquad \text{(where } n \text{ is a positive integer).}$$

The electron eigenfunctions ψ_{nlm} corresponding to the energy E_n are products of the radial wavefunctions R_{nl} and the spherical harmonics Y_{lm}:

$$E_n = -\frac{E_1}{n^2}, \qquad \psi_{nlm}(r,\theta,\phi) = R_{nl}(r)\,Y_{lm}(\theta,\phi),$$

n	l	m	$R_{nl}(r)$	$Y_{lm}(\theta,\phi)$
1	0	0	$2(\frac{1}{a_0})^{3/2}e^{-r/a_0}$	$\frac{1}{2\sqrt{\pi}}$
2	0	0	$(\frac{1}{2a_0})^{3/2}(2-\frac{r}{a_0})e^{-r/2a_0}$	$\frac{1}{2\sqrt{\pi}}$
2	1	0	$(\frac{1}{2a_0})^{3/2}\frac{1}{\sqrt{3}}\frac{r}{a_0}e^{-r/2a_0}$	$\frac{1}{2}\sqrt{\frac{3}{\pi}}\cos\theta$
2	1	±1	$(\frac{1}{2a_0})^{3/2}\frac{1}{\sqrt{3}}\frac{r}{a_0}e^{-r/2a_0}$	$\pm\frac{1}{2}\sqrt{\frac{3}{2\pi}}e^{\pm i\phi}\sin\theta$

Here $a_0 = \hbar^2/m_e e^2$ is the Bohr radius, $E_1 = e^2/2a_0$ is the Rydberg constant, and m_e is the electron mass.

Problem II.3

The differential cross-section $d\sigma/d\Omega$ for the elastic scattering of a particle of mass m from a target characterized by a potential $V(\boldsymbol{r})$ is

$$\frac{d\sigma}{d\Omega} = |f(\boldsymbol{k}, \boldsymbol{k}')|^2, \tag{1}$$

where $\hbar\boldsymbol{k}$ and $\hbar\boldsymbol{k}'$ are the incident and scattered momenta of the particle, with $|\boldsymbol{k}| = |\boldsymbol{k}'|$, and in the first Born approximation, the scattering amplitude $f(\boldsymbol{k}, \boldsymbol{k}')$ can be written as

$$f(\boldsymbol{k}, \boldsymbol{k}') = -\frac{1}{4\pi}\frac{2m}{\hbar^2}\int d^3r\, e^{i(\boldsymbol{k}-\boldsymbol{k}')\cdot\boldsymbol{r}}\, V(\boldsymbol{r}). \tag{2}$$

(a) For a spherically symmetric potential $V(r)$, show that Eq. (2) reduces to

$$f(\theta) = -\frac{2m}{\hbar^2}\frac{1}{q}\int_0^\infty dr\, r\, V(r)\, \sin(qr), \tag{3}$$

where $q = |\boldsymbol{k} - \boldsymbol{k}'|$. Also show that

$$q = 2|\boldsymbol{k}|\sin(\theta/2), \tag{4}$$

where θ is the angle between $\boldsymbol{k}$ and $\boldsymbol{k}'$ (the scattering angle). **[5 points]**

(b) Let $V(r)$ be the Yukawa potential

$$V(r) = \frac{V_0}{\mu}\frac{e^{-\mu r}}{r}. \tag{5}$$

Show that, in this case, Eq. (3) gives

$$f(\theta) = \frac{2mV_0}{\hbar^2\mu}\frac{1}{q^2+\mu^2}. \tag{6}$$

[5 points]

(c) Show that the Yukawa potential (5) reduces to the Coulomb potential in the limit such that $\mu \to 0$ but $(V_0/\mu) \to C$, where C is a constant. Express the constant C in terms of the electric charge Ze of the scattered particle and the charge $Z'e$ that generates the target potential.

Taking the same limit in Eq. (6) and using Eq. (1), derive the Rutherford formula for the differential cross-section $d\sigma/d\Omega$ of Coulomb scattering between the charges Ze and $Z'e$. **[5 points]**

(d) For Coulomb scattering, what is the ratio of $d\sigma/d\Omega$ for scattering in the perpendicular direction ($\theta = \pi/2$) to that in the backward direction ($\theta = \pi$)? Show that this ratio is independent of all parameters of the particle and target. **[5 points]**

(e) If the target is not a point particle, but consists of a spherically symmetric distribution of charge $\rho(r)$, show that the formula in Part (c) is modified to

$$\frac{d\sigma}{d\Omega} = \frac{|F(q)|^2}{q^4}, \tag{7}$$

where $F(q)$ is proportional to the so-called form factor. Derive a formula for $F(q)$ in terms of the charge density distribution $\rho(r)$ and the charge Ze and mass m of the scattered particle. **[5 points]**

Problem II.4

Although dark matter is believed to contribute significantly to the total energy of the universe, the exact nature of the particles that constitute it is unknown. In a large class of theories, dark matter particles are spin 1/2 fermions (denoted here by χ) and the corresponding anti-fermions (denoted here by $\bar{\chi}$), in equal numbers. Since dark matter is electrically neutral, it is possible that these fermions and anti-fermions are not distinct but are in fact the same particle ($\chi \equiv \bar{\chi}$), in which case the dark matter particles are said to be 'Majorana fermions'. In another class of theories the fermions and anti-fermions are distinct particles ($\chi \neq \bar{\chi}$), in which case they are called 'Dirac fermions and anti-fermions'.
Although dark matter is stable, when two dark matter particles collide they may annihilate into lighter particles. Several experiments are underway that attempt to detect these annihilation products. One promising class of experiments involves searching for gamma rays arising from annihilation into $\gamma + \gamma$ and γ + Higgs (H) final states, respectively. In this problem, we consider a way to use the results of such experiments to distinguish between the Majorama and Dirac theories of dark matter.
Assume that dark matter particles are extremely *non*-relativistic (negligible kinetic energy in the frame of the galaxy). For simplicity you may also assume that both the annihilating pair and their annihilation products have relative orbital angular momentum $L = 0$. This means that the space part of their wave function is symmetric under interchange of the particles. The Higgs is a scalar and has zero spin, whereas the photon (γ) is a spin-1 particle. It may help to remember that, when two equal spins are added, the spin part of the wave function with the highest sum ("parallel spins") will be symmetric under intechange of the particles.

(a) Suppose dark matter is Dirac. Show that the process $\chi + \bar{\chi} \to \gamma + H$ is
- forbidden if the Dirac fermions are initially in a singlet state, but
- allowed if they are in a triplet state. **[5 points]**

(b) Under the same assumption consider the process $\chi + \bar{\chi} \to \gamma + \gamma$. What does angular momentum conservation say about this process? What does the exchange symmetry ("statistics") of identical spin 1 particles say? Show that the process is
- forbidden if the Dirac fermions are initially in a triplet state, but
- allowed if they are in a singlet state. **[5 points]**

(c) Now suppose dark matter is Majorana. On the basis of angular momentum and statistics, is the process $\chi + \chi \to \gamma + H$ allowed? **[5 points]**

(d) Under the same assumptions, is the process $\chi + \chi \to \gamma + \gamma$ allowed? **[5 points]**

(e) Find the frequency of a photon emitted in the two processes (annihilation with vs. without Higgs production) in terms of the dark matter mass m_D, the Higgs mass m_H (where $2m_D > m_H$), Planck's constant h and the speed of light. Can the two processes be distinguished on the basis of the observed photon frequency? **[3 points]**

(f) Since the spins of dark matter particles in the universe are randomly oriented, both triplet and singlet initial annihilation states occur. How can the results (a)-(e) be used to distinguish between Dirac and Majorana fermion dark matter candidates? **[2 points]**

Problem II.5

Use the basic assumptions of Cosmology, i.e., that the Universe is homogeneous and isotropic, to consider a universe where conservation of mass and Newton's law of gravity apply. The matter density $\rho(t)$ is constant in space but varies in time due to the action of gravity.
The Hubble parameter $H(t)$ is the fractional rate of expansion of the universe, i.e, $H(t) = \dot{R}/R$. Its value today is $H_0 = 1/t_H$, where $t_H = 13.6 \times 10^9$ years is the Hubble time. Recall that the Gravitational Constant is $G = 6.67 \times 10^{-11}\ \mathrm{m^3kg^{-1}s^{-2}}$.
Pick an arbitrary piece of matter in the universe as the center, for all time t. The motion of a shell of material about this center is described by its time-dependent distance $R(t)$ from the center. Because of homogeneity and isotropy this also represents expansion of the universe as a whole.

(a) Write down the equation of radial motion of a typical particle in the shell at $R(t)$ under Newtonian gravity due to $\rho(t)$. Solve for $\rho(t)$ and express it in terms of the dimensionless deceleration parameter $q(t) = -R\ddot{R}/\dot{R}^2$ and the Hubble parameter $H(t)$. **[5 points]**

(b) Write down the energy integral of your equation of motion from (a), calling the constant of integration the total "energy" E. For a critical universe, i.e., when the total "energy" $E = 0$, what is the value of q? **[5 points]**

(c) Obtain the present value of the critical density ρ_c in kg/m^3. **[5 points]**

(d) Using the cosmological density parameter $\Omega(t) = \rho/\rho_c$, describe the fate of the Universe [long-time behavior of $R(t)$] for $\Omega(t) > 1$, $\Omega(t) = 1$ and $\Omega(t) < 1$. **[5 points]**

(e) For a non-empty Universe with $\rho > 0$, does the Hubble time t_H overestimate or underestimate the age of the Universe? **[5 points]**