UNIVERSITY OF MARYLAND
Department of Physics
College Park, Maryland

Ph.D. PHYSICS QUALIFYING EXAMINATION - PART II

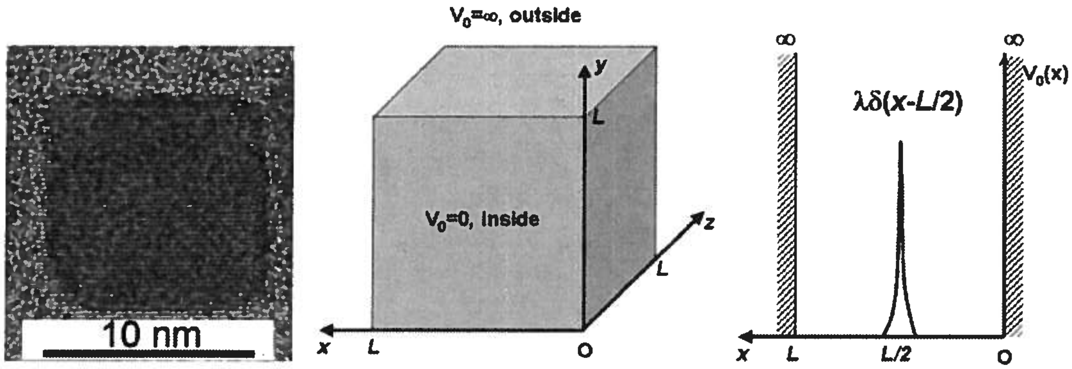
January 21, 2011 **9 a.m. - 1 p.m.**

Do any four problems. Each problem is worth 25 points.

Put all answers on your answer sheets.

Be sure your Qualifier ID Number is at the top right corner of each sheet and turn in solutions to four problems only. If five solutions are turned in we will grade # 1 - # 4.

Problem II.1



A *quantum dot* (see left figure showing a typical transmission electron microscope image of the cross section of a cubic quantum dot) is a structure with dimensions small enough that quantum mechanics plays an important role. Electrons are confined by the boundaries of the quantum dot, and their energy spectrum becomes discrete. The simplest quantum mechanical model for a quantum dot is the "particle-in-a-box" (middle figure), which models the quantum dot as a three-dimensional infinite potential well. Consider an electron with mass m confined inside a cubic quantum dot of size L with the potential

$$V_0(x,y,z) = \begin{cases} 0 & \text{for } \; 0 \le x,y,z \le L, \\ \infty & \text{otherwise.} \end{cases}$$

(a) **[2 points]** Write down the time-independent Schrödinger equation and the boundary conditions on the wavefunction for a single electron in the quantum dot.

(b) **[6 points]** Calculate the energy eigenvalues and eigenfunctions for an electron confined to this quantum dot. Sketch qualitatively the wavefunction of the ground state (i.e. the state with the lowest eigenenergy) along the x direction, at $y{=}z{=}L/2$.

(c) **[4 points]** Design a quantum dot whose characteristic frequency of emission due to transition from the first excited state to the ground state is $\nu = 1$ THz. That is, obtain the size L of the quantum dot that has this property. Give your answer for L in nm. ($\hbar = 1.05 \times 10^{-34}$ Js; $m = 9.11 \times 10^{-31}$ kg; 1 THz = 10^{12} Hz)

(d) **[2 points]** A scattering point defect inside the quantum dot can typically be represented by a repulsive δ function potential. For simplicity, let us consider only a one-dimensional infinite potential square well along x with a point defect located at the center and producing the potential $V_d = \lambda\delta(x - L/2)$ ($\lambda{>}0$), as depicted in the right figure above.

Will the ground state energy increase or decrease after considering the effect of the point defect? Please explain your response.

(e) **[9 points]** In the presence of the point defect, derive an equation for the discontinuity in the derivative of the wavefunction at $x = L/2$. Using this condition, derive a

II.1 (Continued)

transcendental equation for the eigenenergies in the presence of the defect. (Do not try to solve this transcendental equation.)

(f) **[2 points]** Sketch the wavefunction of the ground state with the point defect present and compare with the result obtained in part (b).

Problem II.2

A quantum system has two low-lying energy states, a ground state $|L\rangle$ with energy $E_0 = 0$, and an excited state $|U\rangle$ with energy $E_1 = \mathcal{E}$. There are no other relevant states. The system is perturbed by an interaction Hamiltonian $H' = M\Big(|L\rangle\langle U| + |U\rangle\langle L|\Big)$, where $M \ll \mathcal{E}$.

(a) **[5 points]** Write the total Hamiltonian for this system as a 2×2 matrix in the $|L\rangle$, $|U\rangle$ basis and find its approximate energy eigenvalues $\lambda_{L,U}$, and eigenstates, $|\hat{L}\rangle$ and $|\hat{U}\rangle$ by lowest nontrivial perturbation theory.

(b) **[4 points]** Without calculating the exact eigenvalues, state the sum and the product of the two *exact* energy eigenvalues.

(c) **[4 points]** Now generalize this system to one whose unperturbed ground state is two-fold degenerate at $E_0 = 0$ with orthonormal ground states $|L_i\rangle$ $(i = 1, 2)$. The perturbation H' couples these to the $|U\rangle$ state with equal strengths $M = \langle U|H'|L_1\rangle = \langle U|H'|L_2\rangle$. Write the 3×3 matrix for the total Hamiltonian H.

(d) **[4 points]** Without calculating the individual eigenvectors and eigenvalues, state the sum and the product of the exact eigenvalues for the three exact eigenstates of H.

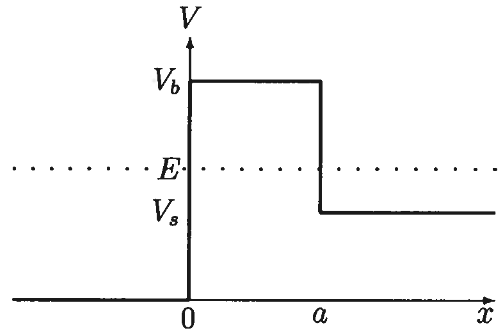
(e) **[4 points]** Calculate the eigenvalues and eigenstates, $|\hat{L}_i\rangle$ and $|\hat{U}\rangle$ corresponding to the $|L_i\rangle$ and $|U\rangle$ to leading order in $M/\mathcal{E}$.

Hint: Use the symmetry of H under the exchange of basis vectors $|L_1\rangle \longleftrightarrow |L_2\rangle$ to simplify your calculations.

(f) **[4 points]** What is the symmetry of each of the three eigenstates in (e) under the $|L_1\rangle \longleftrightarrow |L_2\rangle$ exchange of the basis vectors?

Problem II.3

A beam of electrons is normally incident from vacuum onto a thin film deposited on a substrate. The process can be modeled as a one-dimensional scattering problem involving a potential barrier and a substrate, as shown in the figure. Electrons arrive from the left ($x < 0$) with the energy E, and the barrier is between $x = 0$ and $x = a$ with a constant height $V_b > E$. The potential to the right of the barrier ($x > a$) has a value V_s such that $0 < V_s < V_b$, and we consider the case where $V_s < E < V_b$.



(a) **[3 points]** Write the general solution to the Schrödinger equation in the three regions $x < 0$, $0 < x < a$, and $x > a$ for the transmission problem. (Do not solve for the coefficients of the various terms.)

(b) **[6 points]** In order to simplify the solution, let us model the barrier as a delta function, $V(x) = W_0\delta(x)$. Derive an equation for the discontinuity in the derivative of the wavefunction at the delta-function potential.

(c) **[6 points]** Using this condition, obtain the wavefunction with the energy $E > V_s$ for the transmission problem in the potential

$$V(x) = \begin{cases} 0 & \text{for } x < 0, \\ W_0\delta(x) & \text{for } x = 0, \\ V_s & \text{for } x > 0. \end{cases}$$

(d) **[6 points]** Calculate the transmission coefficient $T = j_{\text{out}}/j_{\text{in}}$, where j is the probability current density, and the subscripts refer to the probability density current past the barrier at $x > 0$ (out) and incident on the barrier at $x < 0$ (in).

(e) **[4 points]** In order for this simple model to be a good approximation to the physical setup, a number of assumptions and/or approximations have to be valid. Discuss one approximation that might have to be corrected in dealing with a real-world experiment.

Problem II.4

(a) **5 points.** Using your knowledge of classical electrodynamics, express the electric and magnetic fields $\boldsymbol{E}(\boldsymbol{r},t)$ and $\boldsymbol{B}(\boldsymbol{r},t)$ in terms of the scalar potential $A^0(\boldsymbol{r},t)$ and the vector potential $\boldsymbol{A}(\boldsymbol{r},t)$ (in the SI system of units).

Consider a solenoid with cross-section area S and a uniform magnetic field B inside (no magnetic field outside of the solenoid). What are the values (in terms of B and S) of the line integral $\oint \boldsymbol{A}\cdot d\boldsymbol{r}$ in the two cases: (i) for a closed loop winding around the solenoid once, (ii) not winding around the solenoid.

Write down an expression (in terms of B and S) for the vector potential $\boldsymbol{A}(\boldsymbol{r})$ outside of the solenoid.

(b) **3 points.** Write the time-dependent Schrödinger equation for a particle with electric charge e, mass m, and zero spin in arbitrary electric and magnetic fields expressed in terms of the scalar and vector potentials A^0 and $\boldsymbol{A}$.

(c) **5 points.** Suppose the wavefunction $\Psi(\boldsymbol{r},t)$ is a solution of the Schrödinger equation with the electromagnetic potentials A^0 and $\boldsymbol{A}$. Consider another wavefunction $\tilde{\Psi}(\boldsymbol{r},t)$ related to the first one by the phase transformation with an arbitrary phase $\varphi(\boldsymbol{r},t)$:

$$\Psi(\boldsymbol{r},t) = e^{i\varphi(\boldsymbol{r},t)}\,\tilde{\Psi}(\boldsymbol{r},t). \qquad (1)$$

Show that wavefunction $\tilde{\Psi}(\boldsymbol{r},t)$ is also a solution of the Schrödinger equation but for different potentials $\tilde{A}^0$ and $\tilde{\boldsymbol{A}}$, and express $\tilde{A}^0$ and $\tilde{\boldsymbol{A}}$ in terms of A^0, $\boldsymbol{A}$, and $\varphi(\boldsymbol{r},t)$. Verify explicitly that the electric and magnetic fields $\boldsymbol{E}$ and $\boldsymbol{B}$ corresponding to $\tilde{A}^0$ and $\tilde{\boldsymbol{A}}$ are the same as those corresponding to A^0 and $\boldsymbol{A}$.

(d) **5 points.** In the rest of the problem, let us omit the time variable t and the scalar potential A^0, and focus on the vector potential $\boldsymbol{A}(\boldsymbol{r})$. In some cases, it is convenient to select $\varphi(\boldsymbol{r})$ so that the transformed vector potential $\tilde{\boldsymbol{A}}$ vanishes, i.e., $\tilde{\Psi}(\boldsymbol{r})$ satisfies the Schrödinger equation with $\tilde{\boldsymbol{A}}(\boldsymbol{r}) = 0$. In order to accomplish this, show that the phase $\varphi(\boldsymbol{r})$ must be selected as follows

$$\varphi(\boldsymbol{r}) - \varphi(\boldsymbol{r}_0) = \frac{e}{\hbar}\int_{\boldsymbol{r}_0}^{\boldsymbol{r}} \boldsymbol{A}(\boldsymbol{r}')\cdot d\boldsymbol{r}', \qquad (2)$$

where $\boldsymbol{r}'$ is the variable of integration, $\boldsymbol{r}_0$ is an arbitrary reference point, and $\varphi(\boldsymbol{r}_0)$ can be set to zero.

In general, the integral (2) may depend on the path of integration, so the phase $\varphi(\boldsymbol{r})$ cannot be defined consistently. However, show that the integral (2) does not depend on the path of integration within a simply-connected region of space if $\boldsymbol{B} = 0$ there.

(e) **7 points.** Consider a double-slit interference experiment illustrated in the figure on the next page. Electrically charged quantum particles propagate from the source S to the detector D at $\boldsymbol{r} = \boldsymbol{r}_D$. They can propagate through either the left or right slit (L or R), which are located symmetrically with respect to the S-D axis.

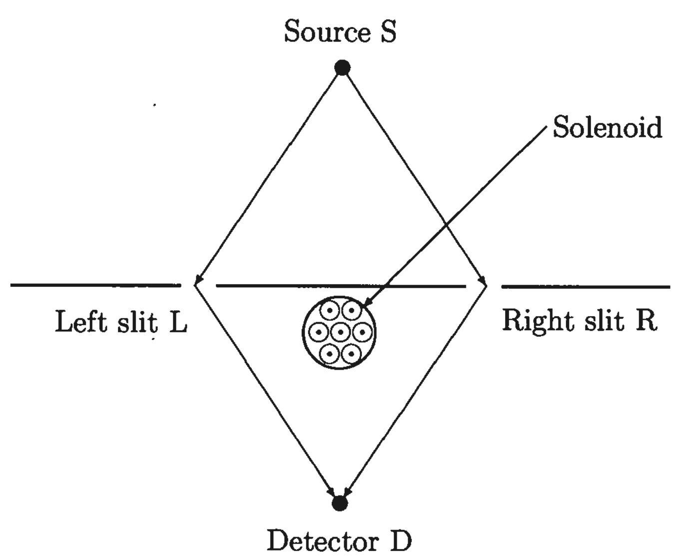
II.4 (Continued)

First ignore the solenoid, i.e. consider the case where $B = 0$ everywhere. The amplitude of the wave function $\Psi = \Psi_L + \Psi_R$ can be written a sum of the two contributions, Ψ_L and Ψ_R, propagating through the left and right slits, respectively. By symmetry, at the detector we can set $\Psi_L(\boldsymbol{r}_D) = \Psi_R(\boldsymbol{r}_D) = \Psi_0$ when $B = 0$.

Now, suppose the solenoid with a non-zero magnetic field B inside the solenoid (drawn as $\odot$) and cross-section area S is placed between the two slits perpendicularly to the plane of the page, as shown in the figure by the $\odot$-decorated circle. There is no magnetic field outside of the solenoid. In the presence of the vector potential $\boldsymbol{A}(\boldsymbol{r})$ created by the solenoid, show that the two contributions to $\Psi(\boldsymbol{r}_D)$ due to propagation through the left and right slits change to $e^{i\varphi_L}\Psi_0$ and $e^{i\varphi_R}\Psi_0$, and find expressions for the phases φ_L and φ_R similar to Eqs. (1) and (2).

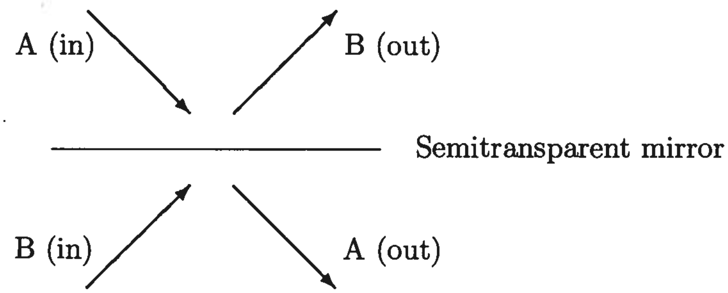
Show that the obtained phase difference $\varphi_L - \varphi_R$ does not depend on how $\boldsymbol{A}(\boldsymbol{r})$ is chosen to characterize the magnetic field of the solenoid and express $\varphi_L - \varphi_R$ in terms of B and S. Does the phase difference depend on the particle attributes e and m and on the distance between the slits?

Calculate the probability $P = |\Psi(\boldsymbol{r}_D)|^2$ of detection of the particles at the detector as a function of the magnetic flux $\Phi = BS$ produced by the solenoid. Show that $P(\Phi)$ is a periodic function and find the period of the function. Make a sketch of $P(\Phi)$.



Problem II.5

Consider a lossless 50/50 beamsplitter, which has two input ports and two output ports and couples two spatial modes of light A and B. The light entering in mode A has a 50% probability of exiting in mode A and a 50% probability of exiting in mode B. The same applies to the light entering in mode B. The beamsplitter can be implemented using a semitransparent mirror, as schematically shown in the figure, but technical details of implementation are not essential for this problem.



We will study the quantum nature of the coupling between these two modes. The two-mode photonic quantum state of the system can be described using the basis $|N_A, N_B\rangle$, where N_A and N_B are the numbers of photons in modes A and B.

(a) **[6 points]** First, consider the case of a single photon, where $N_A + N_B = 1$. Suppose a single photon enters in mode B, so the input wavefunction of the system is $|\psi\rangle_i = |0, 1\rangle$. Write the output wavefunction of the system $|\psi\rangle_o$ after the photon exits from the beamsplitter using the basis of $|0, 1\rangle$ and $|1, 0\rangle$. (The wavefunction $|\psi\rangle_o$ may contain an overall phase factor and a relative phase factor, but you can set these factors to unity by convention.)

(b) **[6 points]** Mathematically, it is convenient to describe the state of N photons using an effective angular momentum formalism (which is unrelated to the actual angular momentum of the photons). Suppose we have $N = N_A + N_B$ photons, and each photon can be in one of the two states A or B. Argue that this system is mathematically equivalent to a system consisting of N spins 1/2, each having the z-projection $+1/2$ or $-1/2$, with the total z-component of the effective angular momentum being $m_J = (N_A - N_B)/2$.

Because the photons are identical Bose particles, their wavefunction must be completely symmetric with respect to their interchange. Show that this requirement implies that the total effective angular momentum in the spin-1/2 description must be $J = (N_A + N_B)/2$.

The mathematical advantage of the effective angular momentum formalism is that the unitary evolution operator of the beamsplitter can be written as the rotation operator $\hat{U} = e^{-i\theta \hat{J}_y}$, where $\hat{J}_y$ is the y-component of the effective angular momentum operator, and θ is an effective rotation angle.

(c) **[7 points]** In the case of a single photon ($N_A + N_B = 1$), we find $J = 1/2$. Thus, the

II.5 (Continued)

evolution operator of the beamsplitter can be written as $\hat{U} = e^{-i\theta\hat{\sigma}_y/2}$, where σ_y is the Pauli matrix acting on the effective two-level system in the basis $|0,1\rangle$ and $|1,0\rangle$.

Write the operator $\hat{U} = e^{-i\theta\hat{\sigma}_y/2}$ explicitly as a linear combination of $\hat{\sigma}_y$ and the unity operator $\hat{I}$. **Hint:** Formally expand the exponential function into a Taylor series and take into account that $\hat{\sigma}_y^2 = \hat{I}$, so that the even and odd terms of the expansion series can be conveniently separated.

Determine the value of the effective rotation angle θ for the 50/50 beamsplitter. **Hint:** When the operator $\hat{U} = e^{-i\theta\hat{\sigma}_y/2}$ is applied on the input state $|\psi\rangle_i = |0,1\rangle$, it must produce the output state $|\psi\rangle_o = \hat{U}|\psi\rangle_i$ that you found in Part (a).

(d) **[6 points]** The value of θ found in Part (c) applies to any number of input photons.

For two photons, the effective angular momentum is $J = 1$, and the basis is $|0,2\rangle$, $|1,1\rangle$, and $|2,0\rangle$. For an input state $|\psi\rangle_i = |1,1\rangle$ with two photons in different modes, determine the output state $|\psi\rangle_o$. What is the probability of finding the output state $|\psi\rangle_o = |1,1\rangle$ where the two photons emerge from the beamsplitter in different modes?

Hint: The angular momentum rotation matrix for $J = 1$ is

$$\hat{U} = e^{-i\theta\hat{J}_y} = \begin{pmatrix} (1+\cos\theta)/2 & \sin\theta/\sqrt{2} & (1-\cos\theta)/2 \\ -\sin\theta/\sqrt{2} & \cos\theta & \sin\theta/\sqrt{2} \\ (1-\cos\theta)/2 & -\sin\theta/\sqrt{2} & (1+\cos\theta)/2 \end{pmatrix}.$$