UNIVERSITY OF MARYLAND
Department of Physics
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# Ph.D. PHYSICS QUALIFYING EXAMINATION - PART II

**January 24, 2014** **9 a.m. - 1 p.m.**

**Do any four problems. Each problem is worth 25 points.**

**Put all answers on your answer sheets.**

**Be sure your Qualifier ID Number is at the top right corner of each sheet and turn in solutions to four problems only. If five solutions are turned in we will grade # 1 - # 4.**

## Problem II.1

The interaction of a two-level atom with a single electromagnetic mode in an optical cavity ("photon" for short, below) can be modeled using the so-called Jaynes-Cummings Hamiltonian:

$$H = \hbar\omega \hat{a}^\dagger \hat{a} + \frac{\hbar\Omega}{2}\sigma_z + \frac{\hbar g}{2}\left(\hat{a}^\dagger \sigma_- + \hat{a}\sigma_+\right)$$

where $\hbar\omega$ is the energy of an optical photon, $\hat{a}^\dagger$ and $\hat{a}$ are the creation and annihilation operators for the photon, $\Omega$ describes the energy splitting of the atom, $g$ accounts for the strength of the coupling between the atom and photon, and the operators

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

act only on the state of the atom. The energy difference between the excited state $\binom{1}{0}$ and the ground state $\binom{0}{1}$ of the atom is $\hbar\Omega$.

(a) **[5 points]** For this part only, assume that $g = 0$. Find all the eigenstates of the atom-photon system and give their energies.

(b) **[5 points]** What is the effect of the term $\frac{\hbar g}{2}\left(\hat{a}^\dagger \sigma_- + \hat{a}\sigma_+\right)$ in the Hamiltonian on an eigenstate you found in part (a)? Clearly explain the physical meaning of this term.

(c) **[5 points]** Now consider the general case, $g \neq 0$. Show that the general ground state is the same as the ground state found in part (a), and that the ground state energy does not depend on $g$.

(d) **[5 points]** Assume that $\omega = \Omega$, and $g$ is non-zero. Show that the two terms of the Hamiltonian, $\hbar\omega \hat{a}^\dagger \hat{a} + \frac{\hbar\omega}{2}\sigma_z$ and $\frac{\hbar g}{2}\left(\hat{a}^\dagger \sigma_- + \hat{a}\sigma_+\right)$, commute.

(e) **[5 points]** Under the assumptions of part (d), and using the fact that the two terms have simultaneous eigenstates, find the allowed energies and eigenstates of the system.

Useful remark: most of the eigenstates in part (a) are degenerate.

## Useful Relations

$$[\hat{a}, \hat{a}^\dagger] = 1 \qquad \hat{a}^\dagger|n\rangle = \sqrt{n+1}\,|n+1\rangle \qquad \hat{a}|n\rangle = \sqrt{n}\,|n-1\rangle$$

## Problem II.2

A spinless particle of mass $M$ and charge $e$ moves in an attractive potential $V(x,y,z) = \frac{k}{2}(x^2+y^2+z^2)$. Let the three quantum numbers of this system be called $n_x$, $n_y$, and $n_z$, and let the natural frequency of the oscillator be called $\omega_0$.

(a) **[3 points]** What are the three lowest energy levels $E_0, E_1, E_2$ and their associated degeneracies?

(b) **[8 points]** Suppose a small perturbing electric field pointing in the $x-$direction of magnitude $E_x = \mathcal{E}\cos(\Omega t)$, with $\Omega \approx \omega_0$, causes transitions among the various oscillator states. Let $n'_x$, $n'_y$, and $n'_z$ specify the final state of the system. Calculate a matrix element which is proportional to the transition amplitudes and that illustrates the selection rules on the quantum numbers.

(c) **[10 points]** Suppose the particle is in the ground state at time $t = 0$. The perturbation will cause a transition into which of the first excited states? Calculate the nonzero amplitude for a transition into this state at later times, to first order in $\mathcal{E}$.

(d) **[2 points]** Calculate the probability for a transition to a first excited state, keeping only the most important term in the expression for the amplitude.

(e) **[2 points]** What is the condition on the electric field strength for the validity of the perturbative calculation of the transition rate?

**Formulae:**

$$\hat{a} = \sqrt{\frac{m\omega_0}{2\hbar}}\left(\hat{x} + \frac{i}{m\omega_0}\hat{p}\right) \tag{1}$$

$$\hat{a}^\dagger = \sqrt{\frac{m\omega_0}{2\hbar}}\left(\hat{x} - \frac{i}{m\omega_0}\hat{p}\right) \tag{2}$$

$$\hat{a}|n\rangle = \sqrt{n}|n-1\rangle \tag{3}$$

$$\hat{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle \tag{4}$$

## Problem II.3

The amplitude for scattering of a quantum particle from a spherically symmetric potential can be expressed in terms of partial waves as

$$f(\theta) = \frac{1}{k} \sum_{\ell=0}^{\infty} (2\ell + 1) e^{i\delta_\ell} \sin \delta_\ell P_\ell(\cos\theta)$$

where $k$ is the particle wave-number.

(a) [**2 points**] Explain what is meant in quantum mechanics by the phase shifts produced by a spherical elastic scatterer and why these are sufficient to specify the scattering properties completely.

(b) [**2 points**] For the case where the potential vanishes outside of a radius $R$, argue that for incident particles of wave number $k$, only partial waves that satisfy $\ell \leq kR$ have significant contributions to the scattering amplitude.

(c) [**2 points**] Write down the condition for which the only significant scattering that occurs is in the s-wave.

(d) [**8 points**] Consider the s-wave scattering from a spherically symmetric finite quantum well potential of constant depth $V$ and radius $R$. What is the relationship between $V$, $R$, particle energy $E$, and $\delta_0$? For the special case, $\delta_0 = \pi/2$ what is the relation?

(e) [**6 points**]

  (i) In the low energy limit, express the energy in terms of the potential depth, $V$, and radius, $R$.

  (ii) In the s-wave approximation what is the scattering cross section?

  (iii) What is the significance of $\delta_0 = \pi/2$?

(f) [**5 points**] What are the conditions for only s-wave scattering to be important with the s-wave phase shift equal to $\pi/2$?

## Problem II.4

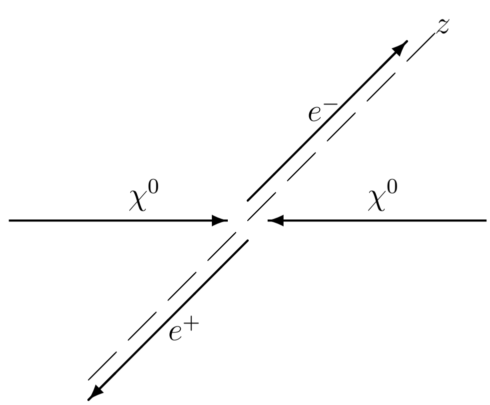
The idea in this problem is to determine selection rules for reactions based on Fermi-Dirac statistics and angular momentum conservation. Consider an electrically neutral fermion (spin-1/2 particle) $\chi^0$, which is its own anti-particle (such a fermion is called "Majorana"). Suppose it annihilates with another $\chi^0$ into an electron ($e^-$) and a positron ($e^+$). Furthermore, assume that the electron has helicity $-1/2$ and the positron has helicity $+1/2$, where helicity $h$ is defined to be the component of the spin of a particle along its direction of motion.

The goal is to prove by contradiction that the initial-state $\chi^0$ pair cannot be in $s$-wave, i.e., it cannot have orbital angular momentum, $L_{initial} = 0$.

(a) **[5 points]** Suppose on the contrary that the initial-state $\chi^0$ pair has $L_{initial} = 0$. Using Fermi-Dirac statistics, determine the allowed value(s) of total spin $S_{initial}$ of this pair.

(b) **[4 points]** Using the above result, determine the allowed value(s) of the total angular momentum $J_{initial}$ of the $\chi^0$ pair.

(c) **[4 points]** Now on to the final state. In the center-of-mass frame of the initial (and therefore the final) state, the $e^-e^+$ are moving in opposite directions along $z$, although they need not be exactly back-to-back (see figure). Given the above helicities of $e^-$ and $e^+$, determine the component of their total spin along their direction of motion, $S_{z\,final}$.




(d) **[4 points]** What is the value of orbital angular momentum $L_{z\,final}$ along the $z$-direction?

(e) **[4 points]** Using the above results, determine the total angular momentum $J_{z\,final}$ along the $z$-direction of motion of $e^-e^+$.

(f) **[4 points]** Therefore, what are the allowed values of total angular momentum $J_{final}$ of $e^-e^+$(*not* just its $z$-component)? Is $s$-wave for the $\chi^0$ pair then allowed?

## Problem II.5

Some of the most intriguing many body properties of bosons are dimensionality dependent. Here you will examine a system of noninteracting bosons in two and three dimensions ($d = 2, 3$).

(a) **[5 points]** Consider $N$ noninteracting spinless bosons of mass $m$ in a volume $V = L^d$. Assume periodic boundary conditions so that the single particle energy levels are eigenstates of momentum $\hbar\mathbf{k}$ with energy $\epsilon(k) = \hbar^2 k^2/2m$. As a function of energy find the density of states $\nu_d(\epsilon)$ for $d = 2, 3$. What is the qualitative difference between $\nu_3(\epsilon \to 0)$ and $\nu_2(\epsilon \to 0)$?

(b) **[6 points]** Now assume a grand canonical description so that the average number of particles, $\langle N \rangle/V$, is fixed and assume that the system size is large so that the thermodynamic or bulk limit is applicable. Obtain integral expressions in $d = 2, 3$ for the number density, $n = \langle N \rangle/V$, in terms of $\nu_d(\epsilon)$ and the Bose-Einstein distribution function

$$n_{BE}(\epsilon) = \frac{1}{e^{(\epsilon-\mu)/k_B T} - 1}, \tag{5}$$

with $\mu$ the chemical potential. What is the maximum value of $\mu$? At fixed temperature is $n(\mu)$ increasing or decreasing as $\mu \to \mu_{max}$?

(c) **[8 points]** In $d = 3$, show that at fixed temperature $n(\mu)$ increases to a finite limit, $n_c$, as $\mu \to \mu_{max}$. In terms of $n$ obtain an expression for the critical temperature, $T_c$, where this occurs. What happens when $n > n_c$ or $T < T_c$?

(d) **[2 points]** Suppose you could measure the momentum distribution in a $d = 3$ Bose gas. What is the experimental signature in the momentum distribution of the critical temperature defined in (c)?

(e) **[4 points]** In $d = 2$ determine $n_c$. What is the corresponding critical temperature? Compare and contrast the momentum distributions in two and three dimensions.

Useful integral:

$$\zeta(3/2) = \frac{4}{\sqrt{\pi}} \int_0^\infty dx \frac{x^2}{\exp(x^2) - 1} = 2.612...$$